Contents

	PRI	EFACE	VII				
1	Inti	Introduction					
	1.1	Definition	1 1				
	1.2	Functional equations	2				
	1.3	Economic dynamics: past and future	3				
	1.4	References	5				
Ι	LI	NEAR DIFFERENCE EQUATIONS	7				
2	Diff	ference Equations: General Principles	9				
	2.1	Definitions	9				
	2.2	Linear difference equations with constant coefficients	11				
		2.2.1 The homogeneous equation	12				
		2.2.2 The non-homogeneous equation	14				
	2.3	Determination of the arbitrary constants	15				
	2.4	References	16				
3	Firs	st-order Difference Equations	19				
	3.1	Solution of the homogeneous equation	19				
	3.2	Particular solution of the non-homogeneous equation	23				
		3.2.1 $g(t)$ is a constant	23				
		3.2.2 $g(t)$ is an exponential function	24				
		3.2.3 $g(t)$ is a polynomial function of degree m	25				
		3.2.4 $g(t)$ is a trigonometric function of the sine-cosine type.	25				
		3.2.5 $g(t)$ is a combination of the previous functions	26				
		3.2.6 The case when $g(t)$ is a generic function of time. Back-					
		ward and forward solutions	26				
	3.3	General solution of the non-homogeneous equation	29				
	3.4	A digression on distributed lags and partial adjustment equa-					
		tions	30				
	3.5	Exercises	33				
		3.5.1 Example	33				
		3.5.2 Other exercises					

XII Contents

	3.6	References	35
4	Firs	t-order Difference Equations in Economic Models	37
	4.1	The cobweb theorem	37
		4.1.1 The cobweb model and expectations	40
		4.1.1.1 The normal price	41
		4.1.1.2 Adaptive expectations	42
	4.2	The dynamics of multipliers	45
		4.2.1 The basic case	45
		4.2.2 Other multipliers	47
		4.2.2.1 A foreign trade multiplier	48
		4.2.2.2 Taxation	49
	4.3	Exercises	50
	4.4	References	53
		Teoreticos I.	
5	Sec	ond-order Difference Equations	55
	5.1	Solution of the homogeneous equation	55
		5.1.1 Positive discriminant $(\Delta > 0)$	56
		5.1.2 Null discriminant $(\Delta = 0)$	57
		5.1.3 Negative Discriminant $(\Delta < 0)$	58
		5.1.4 Stability conditions	60
	5.2	Solution of the non-homogeneous equation	62
	٠	5.2.1 The operational method	63
	5.3	Determination of the arbitrary constants	65
	5.4	Exercises	67
	0.1	5.4.1 Example	67
		5.4.2 Other exercises	70
	5.5	References	71
	5.5	neierences	11
6	Sec	ond-order Difference Equations in Economic Models	73
	6.1	Multiplier-accelerator interaction: the prototype model (Hansen-	-
		Samuelson)	73
		6.1.1 Graphical location of the roots	75
	6.2	Market adjustments and rational expectations	77
	6.3	Hicks' trade cycle model	78
		6.3.1 The workings of the model	83
	6.4	Exercises	88
	6.5	References	90
7	\mathbf{Hig}	her-order Difference Equations	93
	7.1	Solution of the homogeneous equation	93
	7.2	Particular solution of the non-homogeneous equation	94
		7.2.1 The operational method	95
	7.3	Determination of the arbitrary constants	97

Contents XIII

	7.4	Stabil	ity condit	ions	. 97
		7.4.1		ry and sufficient stability conditions	
				son's form)	. 98
		7.4.2		ry and sufficient stability conditions	
				Cohn form)	
	7.5	Exerc			
		7.5.1		e	
		7.5.2		xercises	
	7.6	Refere	ences		. 103
8	Hig	her-or	der Diffe	erence Equations in Economic Models	105
	8.1	Invent	tory cycle	s (Metzler)	. 105
	8.2			s and interaction between the multiplier and	
		the ac	celerator	(Hicks)	. 108
	8.3	Exerc	ises	· · · · · · · · · · · · · · · · · · ·	. 110
	8.4	Refere	ences		. 112
9	Sir	nultan	eous Svs	stems of Difference Equations	113
	9.1		•	2 systems in normal form	. 113
		9.1.1		solution of the homogeneous system: first	
					. 113
		9.1.2		solution of the homogeneous system:	
				(or direct) method	. 116
				Unequal real roots	
				Equal real roots	
				Complex roots	
		9.1.3		ar solution. Determination of the arbitrary	
				ts	
	9.2	First		n systems in normal form	
		9.2.1		natrix solution. The Jordan canonical form .	
		9.2.2	Stability	conditions	. 127
			9.2.2.1		
			9.2.2.2		
		9.2.3	Particul	ar solution	
			9.2.3.1	The operational method	
		9.2.4		ination of the arbitrary constants	
	9.3	Gener		s	
		9.3.1	First-ore	der systems not in normal form	. 138
		9.3.2		order systems	
		515.2	9.3.2.1	An example	
			9.3.2.2	The general case	
			9.3.2.3	Transformation of a higher-order system into	
			0.0.2.0	a first-order system in normal form	. 142
			9.3.2.4	Stability conditions for higher-order systems	
			J.J.Z.T	bushing conditions for ingher order bystems	

XIV Contents

	9.4	Exercises	5
		9.4.1 Example	5
		9.4.2 Other exercises	
	9.5	References	6
10	Sim	ultaneous Difference Systems in Economic Models 14	9
	10.1	Cournot oligopoly	9
	10.2	Multiplier effects in an open economy	2
	10.3	Oligopoly and international trade	5
		10.3.1 The Equilibrium Solution $\dots \dots \dots$	6
		10.3.2 Stability	8
	10.4	Exercises	9
	10.5	References	0
ΙΙ	\mathbf{L}	NEAR DIFFERENTIAL EQUATIONS 16	1
		·	
11		erential Equations: General Principles 16	
		Definitions	
	11.2	Linear differential equations with constant coefficients 16	
		11.2.1 The homogeneous equation	
	11.0	11.2.2 The non-homogeneous equation	
		Determination of the arbitrary constants	
	11.4	References	U
12		t-order Differential Equations 17	
		Solution of the homogeneous equation	
	12.2	Particular solution of the non-homegeneous equation 17	
		12.2.1 $g(t)$ is a constant	
		12.2.2 $g(t)$ is an exponential function	
		12.2.3 $g(t)$ is a polynomial function of degree m 17	
		12.2.4 $g(t)$ is a trigonometric function of the sine-cosine type . 17	
		12.2.5 $g(t)$ is a combination of the previous functions 17	
		12.2.6 $g(t)$ is a generic function of time. The method of variation of parameters	
	12.3	General solution of the non-homogeneous equation	
		Continuously distributed lags and partial adjustment equations 17	
		Exercises	
		12.5.1 Example	
		12.5.2 Other exercises	
	12.6	References	

Contents XV

13	Firs	t-order Differential Equations in Economic Models	185
	13.1	Stability of supply and demand equilibrium	185
	13.2	The neoclassical growth model	191
		13.2.1 Existence of a growth equilibrium	192
		13.2.2 Stability of growth equilibrium	194
		13.2.3 Refinements	197
		13.2.3.1 Depreciation and technical progress	
		13.2.3.2 Golden rule	
		13.2.4 Further developments	200
		13.2.4.1 Adjustment time or, how long is the long run?	200
		13.2.4.2 β -convergence, σ -convergence, and all that	
		13.2.4.3 Endogenous growth	
		Exercises	
	13.4	References	208
14	Seco	ond-order Differential Equations	209
	14.1	Solution of the homogeneous equation	209
		14.1.1 Positive discriminant $(\Delta > 0)$	
		14.1.2 Null discriminant $(\Delta = 0)$	
		14.1.3 Negative discriminant $(\Delta < 0)$	
		14.1.4 Stability conditions	
	14.2	Particular solution of the non-homogeneous equation	
		14.2.1 Variation of parameters	
		General solution of the non-homogeneous equation	
		Determination of the arbitrary constants	
	14.5	Exercises	
		14.5.1 Examples	
		14.5.2 Other exercises	
	14.6	References	221
15	Seco	ond-order Differential Equations in Economic Models	223
		The second-order accelerator	
	15.2	Exercises	226
	15.3	References	228
16	High	her-order Differential Equations	229
		Solution of the homogeneous equation	229
	16.2	Solution of the non-homogeneous equation	231
		16.2.1 Variation of parameters	
	16.3	Determination of the arbitrary constants	
		Stability conditions	
		16.4.1 Necessary and sufficient stability conditions	
		(Routh-Hurwitz)	239

XVI Contents

		16.4.2	Necessary and sufficient stability conditions	0.40
			(Liénard-Chipart)	
	16.5	Exerci	ses	241
		16.5.1	Example	241
			Other exercises	
	16.6	Refere	nces	242
17	Higl	ner-ord	der Differential Equations in Economic Models	243
			ack control and stabilisation policies	
			Introduction	
			Three types of stabilisation policy	
			17.1.2.1 Proportional stabilisation policy	
			17.1.2.2 Mixed proportional-derivative stabilisation pol-	
			icy	
			17.1.2.3 Integral stabilisation policy	
	17.2	Exerci	ses	
			nces	
	1	1001010		201
18	Sim	ultane	ous Systems of Differential Equations	253
	18.1	First-	order 2×2 systems in normal form	253
			General solution of the homogeneous system: first	
			method	254
		18.1.2	General solution of the homogeneous system: second	
			(or direct) method	256
			18.1.2.1 Unequal real roots	
			18.1.2.2 Equal real roots	
			18.1.2.3 Complex roots	
		18.1.3	Particular solution. Determination of the arbitrary	200
		10.1.0	constants	261
	18.2	First o	order $n \times n$ systems in normal form $\dots \dots \dots$	
	10.2		Solution of the homogeneous system	
		10.2.1	18.2.1.1 The matrix exponential and the Jordan canon-	
			ical form	
		1899	Stability conditions	
		10.2.2	18.2.2.1 D-stability, and stabilisation of matrices	
			18.2.2.2 Sensitivity analysis	
			18.2.2.3 A digression on not-wholly-unstable systems.	
			· · ·	
		1002	18.2.2.4 Proof of the stability conditions	
		18.2.3	Particular solution	
		10.0.4	18.2.3.1 Variation of parameters	
	10.0		Determination of the arbitrary constants	
	18.3		al systems	
			First-order systems not in normal form	
		18.3.2	Higher-order systems	287

Contents XVII

		18.3.2.1 An example	. 287
		18.3.2.2 The general case	. 288
		18.3.2.3 Transformation of a higher-order system into	
		a first-order system in normal form	. 289
		18.3.2.4 Stability conditions for higher-order systems	. 292
	18.4	Exercises	. 292
		18.4.1 Example	
		18.4.2 Other exercises	
	18.5	References	. 295
19	Diffe	erential Equation Systems in Economic Models	297
	19.1	Stability of Walrasian general equilibrium of exchange	. 297
		19.1.1 Static stability	. 299
		19.1.2 Dynamic stability	. 301
	19.2	Human capital in a growth model	. 304
		A digression on 'arrow diagrams'	
	19.4	Balanced growth in a multi-sector economy	. 311
	19.5	Exercises	. 316
	19.6	References	. 319
II	I A	ADVANCED TOPICS	323
20	Con	aparative Statics and the Correspondence Principle	325
		Introduction	. 325
		The method of comparative statics	
			. 020
		20.2.1 Purely qualitatively comparative statics	
		20.2.1 Purely qualitatively comparative statics	. 330
	20.3	20.2.2 The inverse comparative statics problem	. 330 . 330
		20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour	. 330 . 330 . 331
		20.2.2 The inverse comparative statics problem	. 330 . 330 . 331 . 334
	20.4	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium	. 330 . 330 . 331 . 334
	20.4	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications	. 330. 330. 331. 334. 336. 338
	20.4	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour	. 330 . 331 . 334 . 336 . 338
	20.420.520.6	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications	. 330 . 331 . 334 . 336 . 338
	20.420.520.6	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour	. 330 . 331 . 334 . 336 . 338 . 343
	20.4 20.5 20.6 20.7	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications	. 330 . 331 . 334 . 336 . 338 . 343
	20.4 20.5 20.6 20.7 20.8	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour	. 330 . 331 . 334 . 336 . 343 . 344 . 345 . 349
21	20.4 20.5 20.6 20.7 20.8 20.9 Stab	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications Extrema and dynamic stability 20.5.1 An application to the theory of the firm Elements of comparative dynamics An illustrative application of the correspondence principle: the IS-LM model Exercises References Collity of Equilibrium: A General Treatment	. 330 . 331 . 334 . 336 . 338 . 343 . 344 . 345 . 349
21	20.4 20.5 20.6 20.7 20.8 20.9 Stat 21.1	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications Extrema and dynamic stability 20.5.1 An application to the theory of the firm Elements of comparative dynamics An illustrative application of the correspondence principle: the IS-LM model Exercises References Collity of Equilibrium: A General Treatment Introduction	. 330 . 331 . 334 . 336 . 338 . 343 . 344 . 345 . 349
21	20.4 20.5 20.6 20.7 20.8 20.9 Stat 21.1	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications Extrema and dynamic stability 20.5.1 An application to the theory of the firm Elements of comparative dynamics An illustrative application of the correspondence principle: the IS-LM model Exercises References Collity of Equilibrium: A General Treatment Introduction Basic concepts and definitions	. 330 . 331 . 334 . 336 . 338 . 343 . 344 . 345 . 349 . 351 . 353
21	20.4 20.5 20.6 20.7 20.8 20.9 Stat 21.1	20.2.2 The inverse comparative statics problem Comparative statics and optimizing behaviour Comparative statics and the dynamic stability of equilibrium 20.4.1 Criticism and qualifications Extrema and dynamic stability 20.5.1 An application to the theory of the firm Elements of comparative dynamics An illustrative application of the correspondence principle: the IS-LM model Exercises References Collity of Equilibrium: A General Treatment Introduction	. 330 . 331 . 334 . 336 . 338 . 343 . 344 . 345 . 349 . 351 . 353

XVIII Contents

		21.2.3 Structural stability	359
	21.3	Qualitative methods: phase diagrams	
		21.3.1 Single equations	
		21.3.2 Two-equation simultaneous systems	
		21.3.2.1 Introduction: phase plane and phase path	
		21.3.2.2 Singular points	
		21.3.2.3 Graphical construction of the trajectories	
		21.3.2.4 Linear systems	
	21.4	Quantitative methods	
		21.4.1 Linearisation	
	21.5	Elements of the qualitative theory of difference equations	
		21.5.1 Single difference equations	
		21.5.2 Two simultaneous difference equations	
		21.5.2.1 Linear systems	
	21.6	Economic applications	
		Exercises	
		References	
22			401
	22.1	General concepts	401
	22.2	The fundamental theorems	402
	22.3	Some economic applications	407
		22.3.1 Global stability of Walrasian general equilibrium	407
		22.3.2 Rules of thumb in business management	414
		22.3.3 Price adjustment and oligopoly under product differ-	
		entiation	415
	22.4	Exercises	419
	22.5	References	420
23			423
	23.1	Preliminary remarks	
		23.1.1 A digression on existence and uniqueness theorems	
	23.2	Some integrable differential equations	
		23.2.1 First-order and first-degree exact equations	426
		23.2.2 Linear equations of the first order with variable coeffi-	
		cients	429
		23.2.3 The Bernoulli equation	
		23.2.4 The Riccati equation	
	23.3	Limit cycles and relaxation oscillations	
		23.3.1 Limit cycles: the general theory	
		23.3.2 Limit cycles: relaxation oscillations	
		23.3.3 Kaldor's non-linear cyclical model	439
		23.3.3.1 The model	439
		23 3 3 2 Kaldor via relayation oscillations	113

Contents XIX

		23.3.3.3 Kaldor via Poincaré's limit cycle 446
23.4	The Lo	otka-Volterra equations
	23.4.1	Construction of the integral curves 453
	23.4.2	Conservative and dissipative systems, and
		irreversibility
	23.4.3	Goodwin's growth cycle
		23.4.3.1 The model
		23.4.3.2 The phase diagram of the model 460
	23.4.4	Palomba's model
		23.4.4.1 The model
		23.4.4.2 Conclusion
23.5	Exercis	ses
		nces
		Theory 473
		action
24.2		ations in continuous time systems
		Codimension-one bifurcations
		The Hopf bifurcation
		Sensitivity analysis and bifurcations: a reminder 484
		Kaldor's non-linear cyclical model again 485
	24.2.5	Oscillations in optimal growth models 486
		24.2.5.1 The model
		24.2.5.2 The optimality conditions 488
		24.2.5.3 Emergence of a Hopf bifurcation 489
		Cycles in an IS-LM model with pure money financing $% \left(1\right) =1$. 491
24.3		ations in discrete time systems
		Codimension-one bifurcations
	24.3.2	The Hopf (or Neimark-Sacker) bifurcation in discrete
		time
		Kaldor's cyclical model in discrete time 498
	24.3.4	Liquidity costs in the firm
		24.3.4.1 The model
		24.3.4.2 The dynamics
		Expectations and multiplier-accelerator interaction 504
24 .4	•	esis and bifurcations
		General
		Dynamical systems
		Economics
		arity-induced bifurcations
24.6	Exercis	ses
24.7	Referen	nces 515

XX Contents

25	Con	nplex Dynamics	519		
	25.1	Introduction			
	25.2	Discrete time systems and chaos			
		25.2.1 The logistic map			
		25.2.2 Intermittency			
		25.2.3 The basic theorems			
		25.2.4 Discrete time chaos in economics			
		25.2.4.1 Chaos in growth theory	. 531		
		25.2.4.2 Exchange rate dynamics and chaos			
	25.3	Continuous time systems and chaos			
		25.3.1 The Lorenz equations, strange attractors, and chaos.			
		25.3.2 Other routes to continuous time chaos			
		25.3.2.1 The Rössler attractor	. 537		
		25.3.2.2 The Shil'nikov scenario	. 538		
		25.3.2.3 The forced oscillator	. 538		
		25.3.2.4 The coupled oscillator	. 539		
		25.3.3 International trade as the source of chaos			
		25.3.4 A chaotic growth cycle			
	25.4	Significance and detection of chaos: Stochastic dynamics or			
		chaos?	. 545		
	25.5	Control of chaos			
		Other approaches			
		25.6.1 Introduction			
		25.6.2 Fast and slow, and synergetics			
		25.6.3 Catastrophe theory			
	25.7	Exercises	. 558		
	25.8	References	. 560		
26	6 Mixed Differential-Difference Equations 5				
		General concepts			
		Continuous vs discrete time in economic models			
		Linear mixed equations			
		The method of solution			
		Stability conditions			
		Approximate methods			
		Delay differential equations and chaos			
	26.8	Some economic applications			
		26.8.1 Kalecki's business cycle model			
		26.8.1.1 The model			
		26.8.1.2 The dynamics	. 585		
		26.8.2 A formalization of the classical price-specie-flow mech-	.		
		anism of balance of payments adjustment			
		26.8.2.1 The model			
		26.8.2.2 Stability	. 591		

Contents XXI

	26.9	Exercises
		References
27		amic Optimization 597
		Introduction
	27.2	Calculus of variations
		27.2.1 Particular cases
		27.2.2 Generalizations
	27.3	The maximum principle
		27.3.1 Statement
		27.3.2 Proof
		27.3.3 Transversality conditions 610
		27.3.3.1 The case with infinite terminal time 611
		27.3.4 Effects of parameter changes on the optimal solution:
		the costate variables
		27.3.5 Discounting
		27.3.6 Particular cases
		27.3.6.1 The bang-bang control case 615
		27.3.6.2 Linear-quadratic problems 616
		27.3.7 The maximum principle in discrete time $\dots \dots 618$
	27.4	Dynamic programming
		27.4.1 Dynamic programming in discrete time: multi-stage
		optimization problems
		27.4.2 Dynamic programming and nonlinear programming 626
		27.4.3 Infinite terminal time 627
		27.4.3.1 Solution by conjecture 628
		27.4.3.2 Solution by iteration
		27.4.3.3 Solution by the envelope theorem 633
		Maximum principle vs. dynamic programming 637
		Exercises
	27.7	References
28	Sade	dle Points and Economic Dynamics 643
		Saddle points in optimal control problems 644
		Optimal economic growth
	20.2	28.2.1 Optimal growth: traditional 644
		28.2.1.1 The setting of the problem 644
		28.2.1.2 The optimality conditions in the basic neo-
		classical model 647
		28.2.1.3 Saddle-point transitional dynamics in the ba-
		sic neoclassical model 651
		28.2.1.4 Optimal and sub-optimal feedback control 653
		28.2.1.4.1 The sub-optimal feedback control
		rule

XXII Contents

	28.2.2	Optimal	growth: endogenous	656
		28.2.2.1	A model of optimal endogenous growth	656
			The conditions for optimal endogenous growth	
			Optimal endogenous growth: saddle-point tran-	
			sitional dynamics	
28.3	Optim	al endoge	nous growth in an open economy	
			Borrower Nation	
			Steady-State Stability and Comparative Dy-	
			namics	671
28.4	Ration	al expect	ations and saddle points	674
			tion	
	28.4.2	Rational	expectations, saddle points, and overshooting	677
		28.4.2.1	A discrete-time equivalent	682
	28.4.3	Rational	expectations and saddle points: the general	
		case		684
28.5	Indete	rminacy a	and sunspots	686
	28.5.1	Indeterm	ninacy and fiscal policy	688
		28.5.1.1	Firms	688
		28.5.1.2	Households	689
		28.5.1.3	Government	689
		28.5.1.4	The optimality conditions	690
		28.5.1.5	The singular point and its nature	693
28.6	Exercis	ses		697
28.7	Refere	nces		699
Bibl	iograp	hy		7 01
Inde	ex			731
Ans	wers to	o Exercis	ses	751