## Contents

1	Some	e Remarks on the Theory of Limit Theorems	
	for N	Iulti-Indexed Sums	1
	1.1	Is It True That All Results for Multi-Indexed Sums Follow	
		From Their Classical Counterparts?	1
	1.2	Is It True That All Classical Methods Apply in the Proofs	
		for Multi-Indexed Sums?	2
	1.3	Is It True That Any Classical Method Can Be Used	
		in the Case of Multi-Indexed Sums?	4
	1.4	Is It True That the Classical Ideas Are Suitable	
		for Multi-Indexed Sums?	5
	1.5	Is It True That the Classical Conditions Are Suitable	
		for Multi-Indexed Sums?	5
	1.6	Is It True That the Classical Constants Do not Change	
		in the Case of Multi-Indexed Sums?	6
	1.7	Some Classical Results Do not Have Counterparts	
		for Multi-Indexed Sums	8
	1.8	Some Results for Multi-Indexed Sums Do not Have	
		Counterparts in the Classical Case	9
	1.9	Some History of the Theory of Limit Theorems	
		for Multi-Indexed Sums	10
		1.9.1 First Stage	10
		1.9.2 Second Stage	12
	1.10	Comments	15
		1.10.1 Books Devoted to Limit Theorems	15
		1.10.2 Monographs Devoted to Random Fields	15



xiii

2	Maximal Inequalities for Multi-Indexed Sums of Independent Random Variables				
	2.1	Estimates for Moments of Multi-Indexed Sums	18		
		2.1.1 The Bahr–Esseen and Dharmadhikari–Jogdeo			
		Inequalities	19		
		2.1.2 Rosenthal's Inequality	21		
		2.1.3 The Marcinkiewicz–Zygmund Inequality	21		
		2.1.4 Skorokhod's Inequality	22		
	2.2	Maximal Inequalities for Distributions of Multi-Indexed Sums	24		
		2.2.1 A Generalization of Petrov's Inequality	24		
		2.2.2 Lévy's Inequality	28		
		2.2.3 Ottaviani's Inequality	30		
		2.2.4 Kolmogorov's Inequality for Probabilities	31		
	2.3	Maximal Inequalities	33		
		2.3.1 A Generalization of Kolmogorov's Inequality			
		for Moments	33		
	2.4	A Generalization of the Hájek–Rényi Inequality	36		
		2.4.1 First Method	36		
		2.4.2 Second Method	36		
		2.4.3 The Hájek-Rényi Inequality for some Classes			
		of Dependent Random Variables	40		
	2.5	Comments	46		
3	Weak Convergence of Multi-Indexed Sums of Independent				
	Rand	dom Variables	49		
	3.1	Main Definitions for Multi-Indexed Sums	50		
	3.2	Limit Laws in the Scheme of Series	51		
		3.2.1 The Case $d = 1$	52		
		3.2.2 The Case of $(d, r)$ -Series $\ldots$	52		
		3.2.3 Centered Sums	54		
	3.3	Conditions for Weak Convergence to a Given Infinitely			
		Divisible Law	55		
	3.4	Weak Convergence of Cumulative Sums	58		
		3.4.1 The Case $d = 1$	58		
		3.4.2 Urbanik Classes	61		
		3.4.3 The Case $d > 1$	62		
		3.4.4 Lévy–Khintchine Classes for $d > 1$	62		
		3.4.5 Feller Classes for $d > 1$	65		
	3.5	Weak Convergence of Cumulative Sums of Identically			
		Distributed Random Variables	66		
	3.6	Comments	75		

4		Law of Large Numbers for Multi-Indexed Sums			
	of In	ndependent Random Variables	77		
	4.1	The Law of Large Numbers in the Scheme of Series	79		
	4.2	The Law of Large Numbers for Cumulative Sums	84		
	4.3	The Law of Large Numbers for Multi-Indexed Sums			
		of Independent Identically Distributed Random Variables	88		
	4.4	The Law of Large Numbers for Multi-Indexed Sums			
		of Independent Identically Distributed Random Variables			
		with the Marcinkiewicz-Zygmund Normalization	89		
	4.5	Comments	98		
5	Almost Sure Convergence of Multi-Indexed Series				
	5.1	Equivalence of Four Types of Convergence	102		
	5.2	The Three Series Theorem for Multi-Indexed Sums	106		
	5.3	The Four Series Theorem	110		
		5.3.1 The Two Series Theorem	111		
		5.3.2 Proofs	112		
	5.4	Convergence of Series of Weighted Independent Identically			
		Distributed Random Variables	116		
	5.5	Essential Convergence	124		
	5.6	Convergence of Multi-Indexed Series on Subsets	125		
	5.7	Convergence of Permutations	126		
	5.8	A Generalization of a Theorem of Chung	128		
	5.9	Comments	130		
6		ndedness of Multi-Indexed Series of Independent			
	Ran	dom Variables	131		
	6.1	Definitions and Auxiliary Results	132		
	6.2	Equivalence of Convergence and Boundedness			
		in Probability of Series of Symmetric Terms	138		
		6.2.1 Sums of Independent Symmetric Random			
		Variables	138		
		6.2.2 Sums of Weighted Independent Identically			
		Distributed Random Variables	141		
	6.3	Skorokhod's Decomposition of a Bounded Series	142		
	6.4	Conditions for the Almost Sure Boundedness			
		of Multi-Indexed Sums	143		
	6.5	Bounded Convergence of Multi-Indexed Series.	146		
	6.6	Moments of the Supremum of Multi-Indexed Sums	147		
	6.7	Comments	153		
7	Rate	e of Convergence of Multi-Indexed Series	155		
	7.1	The Case $d = 1$	155		
	7.2	The Case $d \ge 1$	156		

	7.3	Almost Sure Convergence of All Tails	
		of a Multi-Indexed Series.	157
	7.4	The Doob–Bahr–Esseen Inequality for Tails	161
	7.5	The Hájek-Rényi Inequality for Tails	164
	7.6	The Strong Law of Large Numbers for Tails	
		of a Multi-Indexed Series.	168
		7.6.1 Rate of Convergence for $d = 1, \ldots, \ldots$	170
	7.7	The Law of the Iterated Logarithm for Tails of a Series	171
	7.8	Comments	178
8	The	Strong Law of Large Numbers for Independent	
		dom Variables	179
	8.1	A Necessary Condition for the Strong Law	
		of Large Numbers.	180
	8.2	Generalizations of Kolmogorov's Theorem	181
		8.2.1 The Field $\{b(n)\}$	181
		8.2.2 The Sets $A_1$	181
		8.2.3 The Majorizing Field $\{\lambda(n)\}$	182
		8.2.4 The General Form of the Strong Law	
		of Large Numbers	183
	8.3	Kolmogorov's Strong Law of Large Numbers	186
		8.3.1 Optimality of Kolmogorov's Conditions	187
		8.3.2 Kolmogorov's Condition is not Optimal	189
	8.4	The Marcinkiewicz–Zygmund Strong Law	
		of Large Numbers.	191
		8.4.1 Remarks and Examples	191
	8.5	Chung's Strong Law of Large Numbers	193
		8.5.1 Remarks and Examples	194
	8.6	The Brunk–Prokhorov Strong Law of Large Numbers	197
		8.6.1 Cumulative Sums	197
		8.6.2 Multi-Indexed Sums	202
		8.6.3 Remarks and Examples	205
	8.7	The Teicher–Egorov Strong Law of Large Numbers	210
	8.8	Sets Determining the Strong Law of Large Numbers	214
		8.8.1 Sets Determining the Strong Law	
		of Large Numbers for Multi-Indexed Sums	215
		8.8.2 Some Applications	219
	8.9	Comments	219
9	The	Strong Law of Large Numbers for Independent	
		ntically Distributed Random Variables	223
	9.1	A Generalization of Feller's Theorem	225
	9.2	Proof of the Generalized Feller Theorem	227

	9.3	The Strong Law of Large Numbers for the			
		Marcinkiewicz–Zygmund Normalizing Field	235		
	9.4	The Strong Law of Large Numbers for Indices			
		in a Restricted Domain	246		
		9.4.1 Sectors with Curvilinear Boundaries	247		
		9.4.2 Corollaries to Theorem 9.8.	256		
		9.4.3 Proof of Proposition 9.1	258		
	9.5	Comments	261		
10	The ]	Law of the Iterated Logarithm	265		
	10.1	The Classical Law of the Iterated Logarithm	265		
	10.2	Kolmogorov's Law of the Iterated Logarithm			
		for Multi-Indexed Sums	268		
	10.3	The Law of the Iterated Logarithm for Multi-Indexed Sums			
		of Weighted Independent Identically Distributed			
		Random Variables	276		
	10.4	Corollaries of the Law of the Iterated Logarithm.	293		
		10.4.1 The Chow–Teicher Condition	294		
		10.4.2 The Martikainen Condition	296		
		10.4.3 A Relationship Between Kolmogorov's			
		and Egorov's Conditions	299		
		10.4.4 A Relationship Between Petrov's Condition			
		and the Law of the Iterated Logarithm	301		
	10.5	Chover's Law of the Iterated Logarithm			
		for Multi-Indexed Sums	302		
	10.6	Comments	305 <sub>.</sub>		
11					
	with	Multi-Dimensional Time	307		
	11.1	Some Renewal Theory for $d = 1$	307		
		11.1.1 An Insurance Model	307		
		11.1.2 Asymptotic Behavior of Renewal Functions			
		and Processes	308		
	11.2	<b>-</b>	308		
	11.3	Definition and Properties of Renewal Processes			
		and Functions	310		
	11.4	Asymptotic Behavior of Renewal Functions Constructed			
		from Multi-Indexed Sums	313		
	11.5	The Asymptotic Behavior of Renewal Processes			
		Constructed from Multi-Indexed Sums.	324		
		11.5.1 Duality in the Case $d = 1$	324		
		11.5.2 The Bound $r_0$ for the Rate of Convergence	325		
		11.5.3 The Asymptotic Behavior of Renewal Processes			
		for $d > 1$	326		
	11.6	Comments	331		

12		ence of Moments of Suprema of Multi-Indexed Sums	
		he Strong Law of Large Numbers	333
	12.1	The Existence of Moments of Multi-Indexed Sums	
		for the Marcinkiewicz–Zygmund Normalization	335
	12.2	A Generalized Strong Law of Large Numbers	343
		12.2.1. The Strong Law of Large Numbers in the Space $c_0$	345
	12.3	Moments of the Supremum and the Law	
		of the Iterated Logarithm	347
	12.4	A Generalized Law of the Iterated Logarithm	348
	12.5	Comments	351
13	Com	plete Convergence	353
	13.1	Necessary Conditions for Complete Convergence	355
		13.1.1 Necessary Conditions for Weight Coefficients	355
		13.1.2 Necessary Conditions for Distributions	
		of Random Variables	358
	13.2	Sufficient Conditions for Complete Convergence	361
	13.3	Sufficient Conditions for Complete Convergence	
	1010	of Sums with Random Indices	368
		13.3.1 Gut's Theorem	369
		13.3.2 The Case $\alpha r > 1$	370
		13.3.2 The Case of 2 T   13.3.3 The Baum–Katz Theorem	372
		13.3.4 The Case $\alpha r = 1$	373
	13.4	Complete Convergence of Sums with Non-random Indices	380
	13.4	13.4.1 Series with Repeating Terms	381
		13.4.2 The Case of a Subsequence	383
		13.4.3 The Asmussen–Kurtz Conjecture	384
		13.4.4 Deli's Example.	385
	125		387
	13.5	Complete Convergence of Multi-Indexed Sums	
	12.6	13.5.1 More General Multi-Indexed Series.	390
	13.6	The Asymptotic Behavior of Series of Large Deviation	200
		Probabilities with Respect to a Small Parameter	392
		13.6.1 Asymptotic Behavior of Multi-Indexed Series.	392
		13.6.2 Rate of Convergence in Heyde's Theorem	399
	13.7	Comments	403
Ар	pendix	A: Auxiliary Definitions and Results	405
Ref	erence	S	461
Ind	ex		479