

# Contents

## Book 1: Varieties in Projective Space

<b>1</b>	<b>Basic Notions</b>	3
1	Algebraic Curves in the Plane	3
1.1	Plane Curves	3
1.2	Rational Curves	6
1.3	Relation with Field Theory	9
1.4	Rational Maps	11
1.5	Singular and Nonsingular Points	13
1.6	The Projective Plane	17
1.7	Exercises to Section 1	22
2	Closed Subsets of Affine Space	23
2.1	Definition of Closed Subsets	23
2.2	Regular Functions on a Closed Subset	25
2.3	Regular Maps	27
2.4	Exercises to Section 2	32
3	Rational Functions	34
3.1	Irreducible Algebraic Subsets	34
3.2	Rational Functions	36
3.3	Rational Maps	37
3.4	Exercises to Section 3	40
4	Quasiprojective Varieties	41
4.1	Closed Subsets of Projective Space	41
4.2	Regular Functions	46
4.3	Rational Functions	50
4.4	Examples of Regular Maps	52
4.5	Exercises to Section 4	53
5	Products and Maps of Quasiprojective Varieties	54
5.1	Products	54
5.2	The Image of a Projective Variety is Closed	57
5.3	Finite Maps	60

5.4	Noether Normalisation . . . . .	65
5.5	Exercises to Section 5 . . . . .	65
6	Dimension . . . . .	66
6.1	Definition of Dimension . . . . .	66
6.2	Dimension of Intersection with a Hypersurface . . . . .	69
6.3	The Theorem on the Dimension of Fibres . . . . .	75
6.4	Lines on Surfaces . . . . .	77
6.5	Exercises to Section 6 . . . . .	80
<b>2</b>	<b>Local Properties</b> . . . . .	<b>83</b>
1	Singular and Nonsingular Points . . . . .	83
1.1	The Local Ring of a Point . . . . .	83
1.2	The Tangent Space . . . . .	85
1.3	Intrinsic Nature of the Tangent Space . . . . .	86
1.4	Singular Points . . . . .	92
1.5	The Tangent Cone . . . . .	94
1.6	Exercises to Section 1 . . . . .	95
2	Power Series Expansions . . . . .	98
2.1	Local Parameters at a Point . . . . .	98
2.2	Power Series Expansions . . . . .	100
2.3	Varieties over the Reals and the Complexes . . . . .	104
2.4	Exercises to Section 2 . . . . .	106
3	Properties of Nonsingular Points . . . . .	106
3.1	Codimension 1 Subvarieties . . . . .	106
3.2	Nonsingular Subvarieties . . . . .	110
3.3	Exercises to Section 3 . . . . .	111
4	The Structure of Birational Maps . . . . .	113
4.1	Blowup in Projective Space . . . . .	113
4.2	Local Blowup . . . . .	115
4.3	Behaviour of a Subvariety Under a Blowup . . . . .	117
4.4	Exceptional Subvarieties . . . . .	119
4.5	Isomorphism and Birational Equivalence . . . . .	120
4.6	Exercises to Section 4 . . . . .	123
5	Normal Varieties . . . . .	124
5.1	Normal Varieties . . . . .	124
5.2	Normalisation of an Affine Variety . . . . .	128
5.3	Normalisation of a Curve . . . . .	130
5.4	Projective Embedding of Nonsingular Varieties . . . . .	134
5.5	Exercises to Section 5 . . . . .	136
6	Singularities of a Map . . . . .	137
6.1	Irreducibility . . . . .	137
6.2	Nonsingularity . . . . .	139
6.3	Ramification . . . . .	140
6.4	Examples . . . . .	143
6.5	Exercises to Section 6 . . . . .	146

<b>3</b>	<b>Divisors and Differential Forms</b>	147
1	Divisors	147
1.1	The Divisor of a Function	147
1.2	Locally Principal Divisors	151
1.3	Moving the Support of a Divisor away from a Point	153
1.4	Divisors and Rational Maps	155
1.5	The Linear System of a Divisor	156
1.6	Pencil of Conics over $\mathbb{P}^1$	159
1.7	Exercises to Section 1	161
2	Divisors on Curves	163
2.1	The Degree of a Divisor on a Curve	163
2.2	Bézout's Theorem on a Curve	167
2.3	The Dimension of a Divisor	168
2.4	Exercises to Section 2	169
3	The Plane Cubic	170
3.1	The Class Group	170
3.2	The Group Law	173
3.3	Maps	177
3.4	Applications	179
3.5	Algebraically Nonclosed Field	181
3.6	Exercises to Section 3	183
4	Algebraic Groups	184
4.1	Algebraic Groups	184
4.2	Quotient Groups and Chevalley's Theorem	185
4.3	Abelian Varieties	186
4.4	The Picard Variety	188
4.5	Exercises to Section 4	189
5	Differential Forms	190
5.1	Regular Differential 1-Forms	190
5.2	Algebraic Definition of the Module of Differentials	193
5.3	Differential $p$ -Forms	195
5.4	Rational Differential Forms	197
5.5	Exercises to Section 5	199
6	Examples and Applications of Differential Forms	200
6.1	Behaviour Under Maps	200
6.2	Invariant Differential Forms on a Group	202
6.3	The Canonical Class	204
6.4	Hypersurfaces	206
6.5	Hyperelliptic Curves	209
7	The Riemann–Roch Theorem on Curves	210
7.1	Statement of the Theorem	210
7.2	Preliminary Form of the Riemann–Roch Theorem	213
7.3	The Residue of a 1-Form	217
7.4	Linear Algebra in Infinite Dimensional Vector Spaces	219
7.5	The Residue Theorem	224

7.6	The Duality Theorem . . . . .	225
7.7	Exercises to Sections 6–7 . . . . .	227
8	Higher Dimensional Generalisations . . . . .	229
<b>4</b>	<b>Intersection Numbers . . . . .</b>	<b>233</b>
1	Definition and Basic Properties . . . . .	233
1.1	Definition of Intersection Number . . . . .	233
1.2	Additivity . . . . .	236
1.3	Invariance Under Linear Equivalence . . . . .	238
1.4	The General Definition of Intersection Number . . . . .	242
1.5	Exercises to Section 1 . . . . .	245
2	Applications of Intersection Numbers . . . . .	246
2.1	Bézout's Theorem in Projective and Multiprojective Space . . . . .	246
2.2	Varieties over the Reals . . . . .	248
2.3	The Genus of a Nonsingular Curve on a Surface . . . . .	251
2.4	The Riemann–Roch Inequality on a Surface . . . . .	253
2.5	The Nonsingular Cubic Surface . . . . .	255
2.6	The Ring of Cycle Classes . . . . .	258
2.7	Exercises to Section 2 . . . . .	259
3	Birational Maps of Surfaces . . . . .	260
3.1	Blowups of Surfaces . . . . .	260
3.2	Some Intersection Numbers . . . . .	261
3.3	Resolution of Indeterminacy . . . . .	263
3.4	Factorisation as a Chain of Blowups . . . . .	264
3.5	Remarks and Examples . . . . .	267
3.6	Exercises to Section 3 . . . . .	269
4	Singularities . . . . .	270
4.1	Singular Points of a Curve . . . . .	270
4.2	Surface Singularities . . . . .	273
4.3	Du Val Singularities . . . . .	274
4.4	Degeneration of Curves . . . . .	278
4.5	Exercises to Section 4 . . . . .	281
	<b>Algebraic Appendix . . . . .</b>	<b>283</b>
1	Linear and Bilinear Algebra . . . . .	283
2	Polynomials . . . . .	285
3	Quasilinear Maps . . . . .	285
4	Invariants . . . . .	287
5	Fields . . . . .	288
6	Commutative Rings . . . . .	289
7	Unique Factorisation . . . . .	292
8	Integral Elements . . . . .	293
9	Length of a Module . . . . .	294
	<b>References . . . . .</b>	<b>297</b>
	<b>Index . . . . .</b>	<b>301</b>