

Contents

1	Reading the Black-Scholes Formula in Terms of First and Last Passage Times	1
1.1	Introduction and Notation	1
1.1.1	Basic Notation	1
1.1.2	Exponential Martingales and the Cameron-Martin Formula	2
1.1.3	First and Last Passage Times	2
1.1.4	The Classical Black-Scholes Formula	3
1.2	The Black-Scholes Formula in Terms of First and Last Passage Times	5
1.2.1	A New Expression for the Black-Scholes Formula	5
1.2.2	Comments	6
1.2.3	Proof of Theorem 1.2	7
1.2.4	On the Agreement Between the Classical Black-Scholes Formula (Theorem 1.1) and our Result (Theorem 1.2)	10
1.2.5	A Remark on Theorem 1.2 and Time Inversion	11
1.3	Extension of Theorem 1.2 to an Arbitrary Index ν	13
1.3.1	Statement of the Main Result	13
1.3.2	Some Comments on Theorem 1.3	14
1.3.3	A Short Proof of Theorem 1.3	15
1.4	Another Formulation of the Black-Scholes Formula	16
1.4.1	Statement of the Result	16
1.4.2	First Proof of Theorem 1.4	16
1.4.3	A Second Proof of Theorem 1.4	17
1.5	Notes and Comments	19
2	Generalized Black-Scholes Formulae for Martingales, in Terms of Last Passage Times	21
2.1	Expression of the European Put Price in Terms of Last Passage Times	21
2.1.1	Hypotheses and Notation	21

2.1.2	Expression of $\Pi(K, t)$ in Terms of $\mathcal{G}_K^{(M)}$	22
2.1.3	Proof of Theorem 2.1	22
2.2	Expression of the European Call Price in Terms of Last Passage Times	24
2.2.1	Hypotheses	24
2.2.2	Price of a European Call in Terms of Last Passage Times ...	25
2.2.3	Proof of Theorem 2.2	26
2.3	Some Examples of Computations of the Law of $\mathcal{G}_K^{(M)}$	27
2.4	A More General Formula for the Computation of the Law of $\mathcal{G}_K^{(M)}$..	32
2.4.1	Hypotheses	32
2.4.2	Description of the Law of $\mathcal{G}_K^{(M)}$	33
2.4.3	Some Examples of Applications of Theorem 2.3	34
2.5	Computation of the Law of \mathcal{G}_K in the Framework of Transient Diffusions	37
2.5.1	General Framework	37
2.5.2	A General Formula for the Law of $\mathcal{G}_K^{(X)}$	38
2.5.3	Case Where the Infinitesimal Generator is Given by its Diffusion Coefficient and its Drift	39
2.6	Computation of the Put Associated to a Càdlàg Martingale Without Positive Jumps	41
2.6.1	Notation	41
2.6.2	Computation of the Put Associated to the Martingale $(M_a^{(\nu)}, a \geq 0)$	42
2.6.3	Computation of the Law of $\mathcal{G}_K^{(M^{(\nu)})}$	44
2.6.4	A More Probabilistic Approach of Proposition 2.2	45
2.6.5	An Application of Proposition 2.1 to the Local Times of the Martingale $(\mathcal{E}_t, t \geq 0)$	49
2.7	The case $M_\infty \neq 0$	50
2.7.1	Hypotheses	50
2.7.2	A Generalization of Theorem 2.1	51
2.7.3	First Proof of Theorem 2.5	51
2.7.4	A Second Proof of Theorem 2.5	52
2.7.5	On the Law of $S_\infty := \sup_{t \geq 0} M_t$	53
2.8	Extension of Theorem 2.1 to the Case of Orthogonal Local Martingales	55
2.8.1	Statement of the Main Result	55
2.8.2	First Proof of Theorem 2.6, via Enlargement Theory	56
2.8.3	Second Proof of Theorem 2.6, via Knight's Representation of Orthogonal Continuous Martingales	57
2.8.4	On the Law of $\bigvee_{i=1, \dots, n} \mathcal{G}_{K_i}^{(i)}$	59
2.9	Notes and Comments	63

3	Representation of some particular Azéma supermartingales	65
3.1	A General Representation Theorem	65
3.1.1	Introduction	65
3.1.2	General Framework	66
3.1.3	Statement of the Representation Theorem	66
3.1.4	Application of the Representation Theorem 3.1 to the Supermartingale $(\mathbb{P}(\mathcal{G}_K > t \mathcal{F}_t), t \geq 0)$, when $M_\infty = 0$	67
3.1.5	A Remark on Theorem 3.2	69
3.2	Study of the Pre \mathcal{G}_K - and Post \mathcal{G}_K -processes, when $M_\infty = 0$	70
3.2.1	Enlargement of Filtration Formulae	70
3.2.2	Study of the Post \mathcal{G}_K -Process	71
3.2.3	Study of the Pre \mathcal{G}_K -Process	72
3.2.4	Some Predictable Compensators	73
3.2.5	Expression of the Azéma supermartingale $(\mathbb{P}(\mathcal{G}_K > t \mathcal{F}_t), t \geq 0)$ when $M_\infty \neq 0$	76
3.2.6	Computation of the Azéma Supermartingale	77
3.3	A Wider Framework: the Skorokhod Submartingales	78
3.3.1	Introduction	78
3.3.2	Skorokhod Submartingales	79
3.3.3	A Comparative Analysis of the Three Cases	81
3.3.4	Two Situations Where the Measure \mathbb{Q} Exists	82
3.4	Notes and Comments	87
4	An Interesting Family of Black-Scholes Perpetuities	89
4.1	Introduction	89
4.1.1	A First Example	89
4.1.2	Other Perpetuities	90
4.1.3	A Family of Perpetuities Associated to the Black-Scholes Formula	90
4.1.4	Notation	91
4.1.5	Reduction of the Study	92
4.1.6	Scaling Properties	92
4.1.7	General Case of the Brownian Exponential Martingale of Index $\nu \neq 0$	93
4.1.8	Statement of the Main Results	93
4.2	Proofs of Theorems 4.1, 4.2, 4.3 and 4.4	95
4.2.1	A First Proof of Theorem 4.1	95
4.2.2	Second Proof of Theorem 4.1	96
4.2.3	Proof of Theorem 4.2	98
4.2.4	Proof of Theorem 4.3	99
4.2.5	Proof of Theorem 4.4	102
4.3	Asymptotic Behavior of $\mathbb{E}_1 \left[\exp \left(-\frac{\theta}{2} \Sigma_1 \right) \right]$ as $\theta \rightarrow \infty$	103
4.4	Extending the Preceding Results to the Variables $\Sigma_k^{(\rho, x)}$	106
4.5	Notes and Comments	113

5	Study of Last Passage Times up to a Finite Horizon	115
5.1	Study of Last Passage Times up to a Finite Horizon for the Brownian Motion with Drift	115
5.1.1	Introduction and Notation	115
5.1.2	Statement of our Main Result	116
5.1.3	An Explicit Expression for the Law of $G_x^{(\nu)}(t)$	123
5.2	Past-Future (Sub)-Martingales	127
5.2.1	Definitions	127
5.2.2	Properties and Characterization of PFH-Functions	128
5.2.3	Two Classes of PFH-Functions	131
5.2.4	Another Characterization of PFH-Functions	131
5.2.5	Description of Extremal PFH-Functions	133
5.3	Notes and Comments	141
6	Put Option as Joint Distribution Function in Strike and Maturity	143
6.1	Put Option as a Joint Distribution Function and Existence of Pseudo-Inverses	143
6.1.1	Introduction	143
6.1.2	Seeing $\Pi_M(K, t)$ as a Function of 2 Variables	144
6.1.3	General Pattern of the Proof	144
6.1.4	A Useful Criterion	145
6.1.5	Outline of the Following Sections	145
6.2	The Black-Scholes Paradigm	146
6.2.1	Statement of the Main Result	146
6.2.2	Descriptions of the Probability γ	149
6.2.3	An Extension of Theorem 6.1	155
6.2.4	γ as a Signed Measure on $\mathbb{R}^+ \times \mathbb{R}^+$	157
6.3	Notes and Comments	159
7	Existence and Properties of Pseudo-Inverses for Bessel and Related Processes	161
7.1	Introduction and Definition of a Pseudo-Inverse	161
7.1.1	Motivations	161
7.1.2	Definitions and Examples	162
7.1.3	Aim of this Chapter	163
7.2	Existence of Pseudo-inverses for Bessel Processes	166
7.2.1	Statement of our Main Result	166
7.2.2	A Summary of some Results About Bessel Processes	167
7.2.3	Proof of Theorem 7.1	172
7.2.4	Interpretation in Terms of the Local Martingales ($R_t^{-2\nu}, t \geq 0$)	178
7.3	Some Properties of the r.v.'s ($Y_{x,y}^{(\nu)}, y > x$) ($\nu \geq -\frac{1}{2}$)	179
7.3.1	The Main Theorem	179
7.3.2	Some Further Relations	185

7.4	Two Extensions of Bessel Processes with Increasing Pseudo-Inverses	192
7.4.1	Bessel Processes with Index $\nu \geq -\frac{1}{2}$ and Drift $a > 0$	192
7.4.2	Squares of Generalized Ornstein-Uhlenbeck Processes, also Called CIR Processes in Mathematical Finance	193
7.4.3	A Third Example	195
7.5	The More General Family $(Y_{x,y}^{(\nu,\alpha)}; x < y, \nu \geq 0, \alpha \in [0, 1])$	196
7.5.1	Some Useful Formulae	196
7.5.2	Definition of $(G_y^{(\nu+\theta,\nu)}, y > 0, \nu, \theta \geq 0)$ and $(T_y^{(\nu+\theta,\nu)}, y > 0, \nu, \theta \geq 0)$	197
7.5.3	Existence and Properties of $(Y_{x,y}^{(\nu,\alpha)}; x < y, \nu \geq 0, \alpha \in [0, 1])$	199
7.6	Notes and Comments	201
8	Existence of Pseudo-Inverses for Diffusions	203
8.1	Introduction	203
8.2	Pseudo-Inverse for a Brownian Motion with a Convex, Decreasing, Positive Drift	205
8.3	Study of a Family of \mathbb{R}^+ -Valued Diffusions	210
8.3.1	Definition of the Operator T	210
8.3.2	Study of the Family $(X^{(\alpha)})_{\alpha \geq 0}$	212
8.3.3	Existence of a Pseudo-Inverse when $\alpha = 0$	217
8.4	Existence of Pseudo-Inverses for a \mathbb{R}^+ -Valued Diffusion Started at 0	220
8.4.1	Notations	220
8.4.2	Biane's Transformation	222
8.4.3	Existence of Pseudo-Inverses	225
8.4.4	A Second Proof of Theorem 8.3	228
8.5	Some Consequences of the Existence of Pseudo-Inverses	232
8.5.1	Another Relation Between the Processes X and \bar{X} Started from 0	232
8.5.2	A Time Reversal Relationship	233
8.5.3	Back to the Family $(X^{(\alpha)})_{\alpha \geq 0}$	235
8.6	Notes and Comments	237
A	Complements	239
A.1	Study of the Call Associated to a Strict Local Martingale (see Yen-Yor [93])	239
A.1.1	Introduction	239
A.1.2	Main Results	239
A.1.3	An Extension	241
A.2	Measuring the "Non-Stopping Timeness" of Ends of Previsible Sets (see Yen-Yor, [92])	242
A.2.1	About Ends of Previsible Sets	242
A.2.2	Some Criteria to Measure the NST	242

A.2.3	Computations of Several Examples of Functions $m_L(t)$	244
A.3	Some Connexions with Dupire's Formula	246
A.3.1	Dupire's Formula (see [20, F])	246
A.3.2	Extension of Dupire's Formula to a General Martingale in $\mathcal{M}_+^{0,c}$	246
A.3.3	A Formula Relative to Lévy Processes Without Positive Jumps	248
B	Bessel Functions and Bessel Processes	251
B.1	Bessel Functions (see [46], p. 108-136)	251
B.2	Squared Bessel Processes (see [70] Chapter XI, or [26])	253
B.2.1	Definition of Squared Bessel Processes	253
B.2.2	BESQ as a Diffusion	254
B.2.3	Brownian Local Times and BESQ Processes	254
B.3	Bessel Processes (see [70] Chapter XI, or [26])	255
B.3.1	Definition	255
B.3.2	An Implicit Representation in Terms of Geometric Brownian Motions	256
	References	259
	Further Readings	265
	Index	269