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Appendix: by *Jonathan D. Rogawski*

Modular forms, the Ramanujan conjecture and the

Jacquet-Langlands correspondence

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