

Contents

1	A First Glimpse of Stochastic Processes	1
1.1	Some History	1
1.2	Random Walk on a Line	3
1.2.1	From Binomial to Gaussian	6
1.2.2	From Binomial to Poisson	11
1.2.3	Log–Normal Distribution	13
1.3	Further Reading	15
2	A Brief Survey of the Mathematics of Probability Theory	17
2.1	Some Basics of Probability Theory	17
2.1.1	Probability Spaces and Random Variables	18
2.1.2	Probability Theory and Logic	21
2.1.3	Equivalent Measures	29
2.1.4	Distribution Functions and Probability Densities	30
2.1.5	Statistical Independence and Conditional Probabilities	31
2.1.6	Central Limit Theorem	33
2.1.7	Extreme Value Distributions	35
2.2	Stochastic Processes and Their Evolution Equations	38
2.2.1	Martingale Processes	41
2.2.2	Markov Processes	43
2.3	Itô Stochastic Calculus	53
2.3.1	Stochastic Integrals	53
2.3.2	Stochastic Differential Equations and the Itô Formula	57
2.4	Summary	58
2.5	Further Reading	59
3	Diffusion Processes	63
3.1	The Random Walk Revisited	63
3.1.1	Polya Problem	66
3.1.2	Rayleigh–Pearson Walk	69
3.1.3	Continuous-Time Random Walk	72
3.2	Free Brownian Motion	75

3.2.1	Velocity Process	77
3.2.2	Position Process	81
3.3	Caldeira-Leggett Model	84
3.3.1	Definition of the Model	85
3.3.2	Velocity Process and Generalized Langevin Equation	86
3.4	On the Maximal Excursion of Brownian Motion	90
3.5	Brownian Motion in a Potential: Kramers Problem	92
3.5.1	First Passage Time for One-dimensional Fokker-Planck Equations	94
3.5.2	Kramers Result	97
3.6	A First Passage Problem for Unbounded Diffusion	98
3.7	Kinetic Ising Models and Monte Carlo Simulations	101
3.7.1	Probabilistic Structure	102
3.7.2	Monte Carlo Kinetics	102
3.7.3	Mean-Field Kinetic Ising Model	105
3.8	Quantum Mechanics as a Diffusion Process	110
3.8.1	Hydrodynamics of Brownian Motion	110
3.8.2	Conservative Diffusion Processes	114
3.8.3	Hypothesis of Universal Brownian Motion	115
3.8.4	Tunnel Effect	118
3.8.5	Harmonic Oscillator and Quantum Fields	122
3.9	Summary	126
3.10	Further Reading	127
4	Beyond the Central Limit Theorem: Lévy Distributions	131
4.1	Back to Mathematics: Stable Distributions	132
4.2	The Weierstrass Random Walk	136
4.2.1	Definition and Solution	137
4.2.2	Superdiffusive Behavior	143
4.2.3	Generalization to Higher Dimensions	147
4.3	Fractal-Time Random Walks	150
4.3.1	A Fractal-Time Poisson Process	151
4.3.2	Subdiffusive Behavior	154
4.4	A Way to Avoid Diverging Variance: The Truncated Lévy Flight	155
4.5	Summary	159
4.6	Further Reading	161
5	Modeling the Financial Market	163
5.1	Basic Notions Pertaining to Financial Markets	164
5.2	Classical Option Pricing: The Black-Scholes Theory	173
5.2.1	The Black-Scholes Equation: Assumptions and Derivation	174
5.2.2	The Black-Scholes Equation: Solution and Interpretation	179
5.2.3	Risk-Neutral Valuation	184
5.2.4	Deviations from Black-Scholes: Implied Volatility	189
5.3	Models Beyond Geometric Brownian Motion	191
5.3.1	Statistical Analysis of Stock Prices	192

5.3.2 The Volatility Smile: Precursor to Gaussian Behavior? . . . 205

5.3.3 Are Financial Time Series Stationary? 209

5.3.4 Agent Based Modeling of Financial Markets 214

5.4 Towards a Model of Financial Crashes 221

5.4.1 Some Empirical Properties 222

5.4.2 A Market Model: From Self-organization to Criticality . . . 225

5.5 Summary 233

5.6 Further Reading 234

Appendix A Stable Distributions Revisited 237

A.1 Testing for Domains of Attraction 237

A.2 Closed-Form Expressions and Asymptotic Behavior 239

Appendix B Hyperspherical Polar Coordinates 243

Appendix C The Weierstrass Random Walk Revisited 247

Appendix D The Exponentially Truncated Lévy Flight 253

Appendix E Put–Call Parity 259

Appendix F Geometric Brownian Motion 261

References 265

Index 273