

Contents

Part I Foundations of the Renormalization Group

1	Phase Transitions and the Scaling Hypothesis	5
1.1	Classification of Phase Transitions	5
1.2	Critical Exponents and Universality of Continuous Phase Transitions	9
1.3	The Scaling Hypothesis	12
1.4	Scaling in the Vicinity of Quantum Critical Points	15
	References	21
2	Mean-Field Theory and the Gaussian Approximation	23
2.1	Mean-Field Theory	24
2.1.1	Landau Function and Free Energy	24
2.1.2	Thermodynamic Critical Exponents	26
2.2	Ginzburg–Landau Theory	30
2.2.1	Exact Effective Field Theory	30
2.2.2	Truncated Effective Action: φ^4 -Theory	34
2.3	The Gaussian Approximation	39
2.3.1	Gaussian Effective Action	39
2.3.2	Gaussian Corrections to the Specific Heat Exponent	41
2.3.3	Correlation Function	44
2.3.4	Failure of the Gaussian Approximation in $D < 4$	47
	References	52
3	Wilsonian Renormalization Group	53
3.1	The Basic Idea	54
3.2	Real-Space RG for the One-Dimensional Ising Model	61
3.2.1	Exact Decimation	61
3.2.2	Iteration and Fixed Points of the RG	66
3.2.3	Infinitesimal Form of RG Recursion Relations	69
3.3	General Properties of RG Flows	72
3.3.1	RG Fixed Points and the Critical Surface	72

3.3.2	Local RG Flow Close to a Fixed Point: Classification of Couplings and Justification of the Scaling Hypothesis .	75
3.3.3	Global Properties of RG Flows and Classification of Fixed Points	81
	References	87
4	Critical Behavior of the Ising Model Close to Four Dimensions	91
4.1	Diagrammatic Perturbation Theory	91
4.1.1	Wick Theorem	93
4.1.2	Feynman Diagrams and Linked Cluster Theorem	95
4.1.3	Diagrams for Correlation Functions and the Irreducible Self-Energy	98
4.2	One-Loop Momentum Shell RG	101
4.2.1	Derivation of the RG Flow Equations	102
4.2.2	The Wilson–Fisher Fixed Point	108
4.2.3	Wave Function Renormalization and Anomalous Dimension	113
	References	120
5	Field-Theoretical Renormalization Group	123
5.1	Divergencies and Their Regularization in Field Theory	123
5.2	Perturbative Renormalization	128
5.2.1	The Renormalized Lagrangian	128
5.2.2	Perturbative Calculation of Renormalization Factors	130
5.2.3	Relating Renormalized Perturbation Theory to Experiments	133
5.3	Callan–Symanzik Equation	135
	References	139

Part II Introduction to the Functional Renormalization Group

6	Functional Methods	147
6.1	Generating Functionals for Green Functions	150
6.1.1	Disconnected Green Functions	150
6.1.2	Connected Green Functions	151
6.1.3	Amputated Connected Green Functions	156
6.2	One-Line Irreducible Vertices	159
6.2.1	Generating Functional of the Irreducible Vertices	160
6.2.2	Tree Expansion	163
6.3	Symmetries	167
6.3.1	General Properties	169
6.3.2	Nonrelativistic Particles	171
6.3.3	Dyson–Schwinger and Skeleton Equations	176
	References	179

7	Exact FRG Flow Equations	181
7.1	Cutoffs	181
7.2	Exact FRG Flow Equations for Generating Functionals	186
7.2.1	Disconnected Green Functions	187
7.2.2	Connected Green Functions	188
7.2.3	Amputated Connected Green Functions	189
7.2.4	One-Line Irreducible Vertices	190
7.3	Exact FRG Equations for the Irreducible Vertices	194
7.4	Spontaneous Symmetry Breaking: The Vertex Expansion with Vacuum Expectation Values	200
	References	207
8	Vertex Expansion	209
8.1	Vertex Expansion for Classical ϕ^4 -Theory	210
8.1.1	Exact FRG Flow Equations	212
8.1.2	Rescaled Flow Equations	213
8.1.3	FRG Flow Equations for a Sharp Momentum Cutoff	217
8.2	Recovering the Momentum Shell Results from the FRG	219
8.3	Momentum-Dependent Self-Energy in the Symmetric Phase	221
8.3.1	Scaling Functions	221
8.3.2	Truncation Strategy Based on Relevance	222
8.3.3	FRG Results for the Self-Energy Scaling Function	228
8.4	Momentum-Dependent Self-Energy in the Symmetry-Broken Phase	229
	References	232
9	Derivative Expansion	233
9.1	Derivative Expansion for the $O(N)$ -Symmetric Classical ϕ^4 -Theory	233
9.2	Local Potential Approximation	237
9.2.1	RG Equation for the Effective Potential	237
9.2.2	Fixed Points and Critical Exponents	239
9.3	Beyond the Local Potential Approximation	242
	References	247

Part III Functional Renormalization Group Approach to Fermions

10	Fermionic Functional Renormalization Group	255
10.1	Symmetries of the Two-Fermion Interaction	255
10.2	Exact FRG Flow Equations for the Irreducible Vertices	259
10.2.1	From Superfield to Partially Symmetrized Notation	259
10.2.2	Exact FRG Flow Equations	262
10.2.3	$SU(2)$ -Invariant Flow Equations	268
10.3	Single-Channel Truncations	269

10.3.1	Ladder Approximation in the Particle–Particle Channel . .	269
10.3.2	Random Phase Approximation in the Forward Scattering Channel	273
10.4	Rescaled Flow Equations and Definition of the Fermi Surface . . .	274
10.4.1	Scaling Toward the Fermi Surface	275
10.4.2	Classification of Couplings	282
10.4.3	Exact Integral Equation for the Fermi Surface	284
10.5	One-Loop Patching Approximations	285
10.5.1	Flow of Marginal Couplings	287
10.5.2	Spinless Fermions	290
10.5.3	One-Dimensional g-ology for $SU(2)$ -Invariant Models . .	292
10.5.4	Many-Patch RG for the Square-Lattice Hubbard Model . .	297
	References	302
11	Normal Fermions: Partial Bosonization in the Forward Scattering Channel	305
11.1	Hubbard–Stratonovich Transformation in the Forward Scattering Channel	307
11.2	Exact FRG Flow Equations	310
11.3	Dyson–Schwinger and Skeleton Equations	315
11.4	Ward Identities	317
11.5	Exact Solution of the FRG Flow Equations for Fermions with Linear Dispersion via Ward Identities	322
	References	324
12	Superfluid Fermions: Partial Bosonization in the Particle–Particle Channel	327
12.1	Hubbard–Stratonovich Transformation in the Particle–Particle Channel	328
12.2	Mean-field Approximation and BCS–BEC Crossover	330
12.3	Gaussian Fluctuations	335
12.3.1	Gaussian Effective Action	335
12.3.2	Bosonic Propagators and Order-Parameter Correlations . .	339
12.4	Dyson–Schwinger Equations and Ward Identities	342
12.4.1	Dyson–Schwinger and Skeleton Equations	342
12.4.2	Ward Identities	345
12.5	FRG Approach with Total Momentum Cutoff	346
12.5.1	Superfield Notation	347
12.5.2	Truncation of the Vertex Expansion	351
12.5.3	Truncation with Momentum-Independent Self-Energies . .	357
12.6	Outlook	362
	References	365
	Index	369