

# Contents

<b>1</b>	<b>An Introduction to the Book and a Road Map</b>	<b>1</b>
1.1	Introduction	1
1.2	A Road Map	1
<b>2</b>	<b>An Introduction to the Spectral Method</b>	<b>3</b>
2.1	The Method	3
2.1.1	The Chebyshev Gauss-Lobatto Collocation Differentiation Matrices	4
2.1.2	Why We Use Non-uniform Grids	8
2.1.3	Chebyshev Polynomials and the Pseudo-Spectrum	10
2.1.4	Differentiation Matrices in the Pseudo-Spectral Space	14
2.1.5	Gauss-Radau Collocation Differentiation Matrices	16
2.1.6	Boundary Conditions	17
2.2	What Lies Ahead	17
2.3	Endnotes	18
2.3.1	Accurate Evaluation of Definite Integrals	18
2.3.2	Some Useful Relationships Involving the Chebyshev Polynomials	19
	References	20
<b>3</b>	<b>Steady One-Dimensional (1D) Heat Conduction Problems</b>	<b>21</b>
3.1	One-Domain Problems	21
3.1.1	An Inhomogeneous Problem with Dirichlet-Dirichlet (DD) Conditions	21
3.1.2	An Inhomogeneous Problem with Neumann-Dirichlet (ND) Conditions	23
3.1.3	An Inhomogeneous Problem with Neumann-Neumann (NN) Conditions	24
3.1.4	Homogeneous Problems with Dirichlet and/or Neumann Conditions	26

3.1.5	Robin Boundary Conditions . . . . .	28
3.1.6	Why We are Interested in Homogeneous Differential Problems. . . . .	29
3.1.7	Homogeneous Problems with Radial Coordinates . . . . .	29
3.2	Two-Medium/Two-Domain Problems. . . . .	32
3.2.1	Two-Medium Inhomogeneous Problem with Interface Conditions. . . . .	33
3.2.2	Two-Domain Inhomogeneous Problem and Numerical-Accuracy Considerations . . . . .	36
3.2.3	Homogeneous Problem with Interface Conditions . . . . .	37
3.3	Endnotes . . . . .	38
3.3.1	The Partial Inverse of $A$ and Solvability of $AU = F$ . . . . .	38
3.3.2	Transforming the Generalized Eigenvalue Problem into an Ordinary Eigenvalue Problem . . . . .	42
3.3.3	The Leading Eigenmodes and Non-leading Eigenmodes . . . . .	45
3.3.4	Ellipticity of an Operator . . . . .	46
3.3.5	Inhomogeneous Problem in Polar Coordinates . . . . .	48
3.3.6	Transforming the Two-Domain Coupled Inhomogeneous Problem into a Set of Uncoupled Problems and Extension to the Multi-Domain Case. . . . .	51
3.3.7	Transforming the Two-Domain Coupled Homogeneous Problem into an Ordinary Eigenvalue Problem . . . . .	54
3.3.8	The Distinction Between the Error and the Residual . . . . .	57
3.3.9	A Particular Caution When Solving Coupled Differential Problems. . . . .	58
	References . . . . .	60
<b>4</b>	<b>Unsteady 1D Heat Conduction Problems. . . . .</b>	<b>61</b>
4.1	An Inhomogeneous Problem with Neumann Conditions . . . . .	61
4.1.1	Continuous Problem . . . . .	62
4.1.2	Time Discretization . . . . .	62
4.1.3	Fully Discretized System . . . . .	64
4.1.4	Note on the Flux Solvability Condition . . . . .	64
4.2	An Inhomogeneous Problem with a Non-linear Source Term . . . . .	65
4.3	A 1D Convection-Diffusion Heat Equation . . . . .	65
4.4	Some Closing Thoughts on Time Marching . . . . .	66
4.5	Endnotes . . . . .	67
4.5.1	From Taylor Expansions to Time Discretization . . . . .	67
4.5.2	The Destiny of a Solution to a Transient Problem Whose Steady Counterpart is not Solvable. . . . .	68
4.5.3	Comment About the Accuracy of the Time Discretization Approach. . . . .	69

4.5.4	Time Integration can be Spectrally Accurate . . . . .	71
	References . . . . .	73
<b>5</b>	<b>Steady Two-Dimensional (2D) Heat Conduction Problems . . . . .</b>	<b>75</b>
5.1	One-Domain Problems . . . . .	75
5.1.1	Inhomogeneous Problem in Cartesian Coordinates . . . . .	75
5.1.2	Inhomogeneous Problems in $(r, z)$ Cylindrical Coordinates . . . . .	85
5.1.3	Homogeneous Problems . . . . .	89
5.2	Two-Medium Inhomogeneous Problems with Interface Conditions . . . . .	90
5.3	Endnotes . . . . .	94
5.3.1	Kronecker Product . . . . .	94
5.3.2	Multi-dimensional Matrix Multiplication . . . . .	95
5.3.3	An Optimized Solver for $\nabla^2 u = f$ Using Successive Diagonalization . . . . .	97
5.3.4	Solving the 2D Two-Medium Inhomogeneous Diffusion Problems by Successive Diagonalization . . . . .	105
	References . . . . .	111
<b>6</b>	<b>2D Closed Flow Problems: The Driven Cavity . . . . .</b>	<b>113</b>
6.1	The Driven Cavity and Navier-Stokes Equations . . . . .	113
6.2	The Stokes Problem . . . . .	114
6.3	Outline of the Remaining Part of the Chapter . . . . .	116
6.4	The 2D Stokes Eigenproblem . . . . .	116
6.4.1	Numerical Procedure . . . . .	117
6.4.2	Spurious Pressure Modes (SPM) . . . . .	120
6.5	The Stokes Steady Flow Problem . . . . .	123
6.5.1	Regularizing the Model . . . . .	124
6.5.2	Numerical Procedure . . . . .	124
6.6	The Unsteady Stokes Flow Problem . . . . .	125
6.6.1	The $(\mathbf{v}, p)$ Uncoupling by Projection-Diffusion . . . . .	126
6.6.2	Time Discretization . . . . .	129
6.6.3	Space Discretization of the Projection Step . . . . .	130
6.6.4	A Comment on Why We Do not Use $\nabla^2 p$ to Determine the Pressure Field . . . . .	139
6.6.5	Other Stokes Solvers . . . . .	139
6.7	The Navier-Stokes Flow Problem . . . . .	140
6.7.1	The Non-linear Equations . . . . .	140
6.7.2	Numerical Procedure . . . . .	140
6.7.3	Numerical Stability Considerations . . . . .	141
6.8	Endnotes . . . . .	143
6.8.1	Ellipticity of the Stokes Operator . . . . .	143
6.8.2	Constructing the 2D/3D Pressure Operator from Projection-Diffusion . . . . .	144

6.8.3	The Projection-Diffusion “toy” Problem Posed in a Two-domain Configuration . . . . .	148
6.8.4	Projection-Diffusion in 2D Multi-domain. . . . .	156
	References . . . . .	169
<b>7</b>	<b>Applications to Transport Instabilities . . . . .</b>	<b>171</b>
7.1	Ignition in a Solid . . . . .	173
7.1.1	Method (a): Time Marching . . . . .	174
7.1.2	Method (b): Steady-Solution Determination and Continuation Methods . . . . .	177
7.1.3	Method (c): The Leading Eigenvalue of the Problem Linearized About a Base State . . . . .	183
7.1.4	Some Closing Thoughts . . . . .	185
7.2	The Rayleigh-Bénard Problem in a Porous Medium . . . . .	185
7.2.1	Scaled Equations. . . . .	187
7.2.2	Method (a): Time Marching with the Non-linear Problem . . . . .	188
7.2.3	Method (b): Steady-Solution Determination . . . . .	193
7.2.4	Method (c): The Leading Eigenvalue of the Problem Linearized About a Base State . . . . .	199
7.2.5	Method (d): The Time Marching with the Problem Linearized About a Base State . . . . .	203
7.3	Solidification Front . . . . .	204
7.3.1	The Model and Its Base State. . . . .	205
7.3.2	The Front Instability and the Linearized Equations . . . . .	206
7.3.3	The Numerical Evaluation of the Time Growth Rate, $\sigma$ . . . . .	209
7.3.4	Results of the Calculation . . . . .	210
	References . . . . .	211
<b>8</b>	<b>Exercises for the Reader . . . . .</b>	<b>213</b>
	References . . . . .	226
	<b>Index . . . . .</b>	<b>227</b>