

Contents

1 Preliminaries	1
1.1 Abstract Tools	2
1.1.1 The Lax-Milgram Theorem	2
1.1.2 Galerkin's Method	3
1.1.3 Polars, Projections, and Pseudo Inverses	4
1.1.4 Saddle Point Problems	7
1.2 The Potential Equation: Variational Framework and Finite Elements	8
1.2.1 Sobolev Spaces	8
1.2.2 The Potential Equation	18
1.2.3 The Finite Element Method	23
1.2.4 Solvers for Linear Systems	30
1.2.5 The Pure Neumann Problem	34
1.2.6 The Schur Complement System	36
1.2.7 The Exterior Laplace Problem	39
1.3 Boundary Integral Equations	43
1.3.1 Surface Potentials and Representation Formulae	43
1.3.2 Boundary Integral Operators and Equations	45
1.3.3 The Exterior Laplace Problem in Two Dimensions	48
1.3.4 Representations of Steklov-Poincaré Operators	49
1.3.5 Newton Potentials	50
1.3.6 Generalizations to the Potential Equation	51
1.3.7 The Boundary Element Method	52
1.3.8 BEM Approximation of S^{int} , S^{ext} , and $\tilde{N}$	56
2 One-Level FETI/BETI Methods	63
2.1 Skeleton Formulations	64
2.1.1 Continuous Skeleton Formulation	64
2.1.2 Discrete Skeleton Formulations	66
2.1.3 Error Analysis of the Discrete Skeleton Formulation	69
2.1.4 Conditioning of the Skeleton Problem	70

2.2	Formulation of One-Level FETI/BETI Methods	71
2.2.1	Formulation of Classical FETI/BETI	72
2.2.2	All-Floating (Total) FETI/BETI	84
2.2.3	FETI/BETI in Terms of Fluxes and Traces	87
2.2.4	Preconditioning	89
2.2.5	Implementation Issues	94
2.2.6	Inexact FETI/BETI Methods	100
2.3	Balancing Neumann-Neumann Methods	101
2.4	Introduction to the Analysis of FETI/BETI	104
2.4.1	The Unpreconditioned Case	104
2.4.2	Abstract Framework for the Preconditioned Case	107
2.5	Technical Tools	113
2.5.1	Globs and Cut-Off Functions	113
2.5.2	Regularity Assumptions on the Subdomains	115
2.5.3	An Explicit Poincaré Inequality	116
2.5.4	Trace and Poincaré Inequalities for FE Functions	119
2.5.5	Cut-Off Estimates	122
2.5.6	An Explicit Sobolev Extension	126
2.5.7	A Subdomain Transfer Operator for FE Functions	129
2.5.8	Uniform Bounds for the Constants c_0 and c_K	130
2.5.9	An Elementary Inequality	133
2.6	Preconditioner Analysis for Subdomain Resolved Coefficients	134
2.6.1	An Energy Estimate of P_D	135
2.6.2	The Case $Q = M_{SD}^{-1}$	137
2.6.3	Diagonal Choice of Q	137
2.6.4	Alternative Scalings	142
2.7	Numerical Results	144
2.8	Other PDEs and Other Discretization Spaces	146
2.8.1	Other Discretizations Spaces for H^1 -Problems	146
2.8.2	FETI and Balancing Type Methods for General PDEs	150
3	Multiscale Problems	157
3.1	Introduction and Notation	158
3.1.1	Multiscale Problems	158
3.1.2	The Coefficient Model	159
3.1.3	Notation	160
3.2	A Rough Condition Number Bound	161
3.3	Towards a Refined Condition Number Bound	163
3.3.1	Assumptions on the Coefficient	164
3.3.2	Choice of Scalings	166
3.3.3	Technical Tools	167
3.3.4	Weighted Poincaré Inequalities: Introduction	170
3.3.5	Semi-abstract Condition Number Bounds	171

3.4	Weighted Poincaré Inequalities (WPI)	182
3.4.1	Regularity Assumptions on the Weight Function	183
3.4.2	Weighted Poincaré Inequalities in H^1	184
3.4.3	Weighted Poincaré Inequalities for FE Functions	187
3.4.4	Dependence on Geometrical Parameters	189
3.5	Robust FETI Condition Number Bounds Explicit in η	203
3.5.1	Constant Coefficients in the Boundary Layers	204
3.5.2	Quasi-Monotone Coefficients in the Boundary Layers	205
3.5.3	Condition Number Bounds Using Artificial Coefficients	208
3.6	Numerical Examples and Conclusions	212
4	Unbounded Domains	215
4.1	Model Problem and Skeleton Formulation	216
4.1.1	Model Problem	216
4.1.2	Skeleton Formulation	217
4.1.3	FETI/BETI Formulation	219
4.2	Analysis	222
4.2.1	Assumptions on the Subdomains and on the Data	222
4.2.2	Auxiliary Meshes, Globbs, and Cut-Off Functions	224
4.2.3	A Condition Number Estimate Involving H_0	228
4.2.4	An Alternative Condition Number Bound for Classical FETI/BETI	232
4.2.5	Further Remarks on All-Floating FETI/BETI	240
4.2.6	The Case of Two Dimensions	241
4.3	Numerical Results	242
4.4	Implementation Issues	245
5	Dual-Primal Methods	247
5.1	Formulation of FETI/BETI-DP Methods	248
5.1.1	Dual-Primal Spaces	249
5.1.2	Saddle Point Formulation and Dual-Primal Method	254
5.1.3	Preconditioning	255
5.1.4	The BDDC Method	258
5.2	Analysis of FETI/BETI-DP and BDDC	259
5.2.1	An Abstract Condition Number Bound	259
5.2.2	Estimates for the P_D Operator	260
5.2.3	Main Result	264
5.2.4	Extensions to Multiscale Coefficients	265
5.2.5	An Estimate for the BDDC Preconditioner	267
5.3	Implementation of FETI/BETI-DP Methods	268
5.3.1	Splitting Into Primal and Dual Subspaces	270
5.3.2	Parametrization of $\tilde{W}$ and $\tilde{W}^*$	273
5.3.3	Realization of $\tilde{T}$ and $\tilde{T}^T$	275

5.3.4 Realization of $\widetilde{S}^{-1}$ 276

5.3.5 Comparison of Dual-Primal and One-Level Methods 279

5.4 Inexact Solvers, Multilevel Extensions, and Deflation 281

References..... 283

List of Symbols 303

Index..... 311