Contents

Pr	eface		vii
No	otation		ix
I	Clas	sical Theory	
1	Preli	minaries	3
	1.1	Basic Differential Operators	3
	1.2	Linear and Quasilinear Partial Differential Equations	5
	1.3	Solutions of Some Particular Equations	8
	1.4	Boundary Value Problems	10
		1.4.1 Boundary Value Problems for Poisson's Equation	10
		1.4.2 Boundary Value Problems for the Heat Equation1.4.3 Boundary Value Problems for the Wave Equation	11 12
2	Dowt!	•	13
Z		al Differential Equations and Mathematical Modeling	
	2.1	Conservation Laws: Continuity Equations	13
	2.2	Reaction-Diffusion Systems	16
	2.3	The One-Dimensional Wave Equation	17
	2.4	Other Equations in Mathematical Physics	18
3	Ellip	tic Boundary Value Problems	21
	3.1	Green's Formulas	21
	3.2	The Fundamental Solution of Laplace's Equation	22
	3.3	Mean Value Theorems for Harmonic Functions	25
	3.4	The Maximum Principle	26
	3.5	Uniqueness and Continuous Dependence on Data for the Dirichlet Problem	29
	3.6	Green's Function of the Dirichlet Problem	30
	3.7	Poisson's Formula	31
	3.8	Dirichlet's Principle	34
	\mathcal{L}_{i}		_



	3.9	The Generalized Solution of the Dirichlet Problem	37
	3.10	Abstract Fourier Series	42
	3.11	The Eigenvalues and Eigenfunctions of the Dirichlet Problem	45
	3.12	The Case of Elliptic Equations in Divergence Form	50
	3.13	The Generalized Solution of the Neumann Problem	51
	3.14	Complements 3.14.1 Harnack's Inequality 3.14.2 Hopf's Maximum Principle 3.14.3 The Newtonian Potential 3.14.4 Perron's Method	55 55 57 59 62
		3.14.5 Layer Potentials	68
		3.14.6 Fredholm's Method of Integral Equations	70
	3.15	Problems	71
4	Mixe	d Problems for Evolution Equations	87
	4.1	The Maximum Principle for the Heat Equation	87
	4.2	Vector-Valued Functions	90
	4.3	The Cauchy–Dirichlet Problem for the Heat Equation	91
	4.4	The Cauchy–Dirichlet Problem for the Wave Equation	99
	4.5	Problems	102
5	The (Cauchy Problem for Evolution Equations	109
	5.1	The Fourier Transform	109 109 110 112
	5.2	The Cauchy Problem for the Heat Equation	116
	5.3	The Cauchy Problem for the Wave Equation	119
	5.4	Nonhomogeneous Equations: Duhamel's Principle	123
	5.5	Problems	125
II		dern Theory	
6	Distr	ibutions	131
	6.1	The Fundamental Spaces of the Theory of Distributions	131
	6.2	Distributions: Examples; Operations with Distributions	133

Contents xiii

		6.2.1 Regular Distributions	133
		6.2.2 The Dirac Distribution	134
		6.2.3 Differentiation	134
		6.2.4 Multiplication by a Smooth Function	136
		6.2.5 Composition with a Smooth Function	137
		6.2.6 Convolution	137
		6.2.7 Distributions of Compact Support	139 142
	6.2	· ·	142
	6.3	The Fourier Transform of Tempered Distributions 6.3.1 The Fourier Transform on $\mathcal{S}'(\mathbf{R}^n)$	142
		6.3.2 The Fourier Transform on $L^2(\mathbb{R}^n)$	144
		6.3.3 Convolution in 8'	144
	6.4	Problems	145
7	Sobol	ev Spaces	149
	7.1	The Sobolev Spaces $H^m(\Omega)$	149
	7.2	The Extension Operator	152
	7.3	The Sobolev Spaces $H_0^m(\Omega)$	156
	7.4	Sobolev's Continuous Embedding Theorem	159
	7.5	Rellich-Kondrachov's Compact Embedding Theorem	163
	7.6	The Embedding of $H^m(\Omega)$ into $C(\overline{\Omega})$	165
	7.7	The Sobolev Space $H^{-m}(\Omega)$	167
	7.8	Fourier Series in $H^{-1}(\Omega)$	172
	7.9	Generalized Solutions of the Cauchy Problems	175
8	The V	Variational Theory of Elliptic Boundary Value Problems	180
	8.1	The Variational Method for the Dirichlet Problem	180
	8.2	The Variational Method for the Neumann Problem	184
	8.3	Maximum Principles for Weak Solutions	186
	8.4	Regularity of Weak Solutions	191
	8.5	Regularity of Eigenfunctions	198
	8.6	Problems	201
III	Se	milinear Equations	
9		linear Elliptic Problems	208
	9.1	•	208

xiv	Contents

	9.2	Application of Banach's Fixed Point Theorem	211
	9.3	Application of Schauder's Fixed Point Theorem	213
	9.4	Application of the Leray–Schauder Fixed Point Theorem	215
	9.5	The Monotone Iterative Method	218
	9.6	The Critical Point Method	220
	9.7	Problems	225
1Ó	The	Semilinear Heat Equation	227
	10.1	The Nonhomogeneous Heat Equation in $H^{-1}(\Omega)$	227
	10.2	Regularity Results	233
	10.3	Application of Banach's Fixed Point Theorem	238
	10.4	Application of Schauder's Fixed Point Theorem	241
	10.5	Application of the Leray-Schauder Fixed Point Theorem	245
11	The	Semilinear Wave Equation	248
	11.1	The Nonhomogeneous Wave Equation in $H^{-1}\left(\Omega\right)$	248
	11.2	Application of Banach's Fixed Point Theorem	252
	11.3	Application of the Leray-Schauder Fixed Point Theorem	257
12	Sem	ilinear Schrödinger Equations	262
	12.1	The Nonhomogeneous Schrödinger Equation	262
	12.2	Properties of the Schrödinger Solution Operator	266
	12.3	Applications of Banach's Fixed Point Theorem	268
	12.4	Applications of Schauder's Fixed Point Theorem	272
Bib	liogra	aphy	275
Index			278