

Contents

1	Why Mathematics?	1
1.1	What Is Mathematics?	2
1.2	A Historical Perspective	2
1.3	On the Superiority of Formal Reasoning	4
1.4	The Mathematics of Computing	4
1.4.1	The Analysis of Programs	5
1.4.2	Formal Specification of Requirements	6
1.4.3	Reasoning About Programs	6
1.4.4	The Limits of Computing	7
1.4.5	A Physical Embodiment of Mathematics	8
1.5	Terminology	8
2	Propositional Logic	9
2.1	What Is a Proposition?	9
2.2	The Language of Discourse	10
2.3	Model Theory	15
2.3.1	Truth Tables: Propositional Operators	16
2.3.2	Properties of Propositional Sentences	22
2.3.3	Conclusion	28
2.4	Proof Theory	29
2.4.1	What Is a Proof?	29
2.4.2	Example Axioms and Rules of Inference	30
2.4.3	Why Proofs?	47
2.5	Comparing Truth Tables and Proofs	47
2.5.1	Completeness	48
2.5.2	Soundness	48
2.6	More About Propositional Logic	48
2.6.1	Boolean Algebra	48
2.6.2	Doing with Less	50
2.6.3	Doing with Even Less	51
2.6.4	What if the Axioms Are Wrong?	52

2.7	Some Notes	53
2.7.1	Notation and Symbols	53
2.7.2	Proof Elegance	54
2.7.3	How to Come up with a Proof	55
2.7.4	Not All Proofs Are Equal	55
3	Predicate Calculus	57
3.1	The Limits of Propositional Logic	57
3.2	The Language of Discourse	58
3.3	The Use of Variables	61
3.3.1	Free Variables	62
3.3.2	Substitution of Variables	64
3.4	Axiomatisation of Predicate Logic	67
3.4.1	Universal Quantification	67
3.4.2	Existential Quantification	69
3.4.3	Existential Quantification and Equality	76
3.5	Beyond Predicate Calculus	76
4	Sets	79
4.1	What Are Sets?	79
4.2	Queries About Sets	80
4.3	Comparing Sets	81
4.3.1	Subsets	81
4.3.2	Set Equality	83
4.4	Constructing Sets	84
4.4.1	Finite Sets	84
4.4.2	The Empty Set	85
4.4.3	Set Comprehensions	86
4.5	Set Operators	90
4.5.1	Set Union	92
4.5.2	Set Intersection	93
4.5.3	Set Complement	95
4.5.4	Set Difference	97
4.5.5	Other Properties of the Set Operators	98
4.5.6	Generalised Set Operators	99
4.6	Sets as a Boolean Algebra	102
4.7	More About Types	104
4.7.1	Types and Sets	104
4.7.2	On Why Types Are Desirable: Russell's Paradox	108
4.8	Summary	110
5	Relations	111
5.1	An Informal View of Relations	111
5.2	Formalising Relations	112
5.2.1	The Type of a Relation	113
5.2.2	Some Basic Relations	114

5.2.3	Relational Equality	115
5.2.4	Combining Relations as Sets	115
5.3	Domain and Range	118
5.4	Building New Relations from Old Ones	119
5.4.1	Composition of Relations	119
5.4.2	Relational Inverse	123
5.4.3	Repeated Relations	127
5.4.4	Closure	131
5.5	Other Relational Operators	134
5.6	Beyond Binary Relations	138
5.7	Summary	139
6	Classifying Relations	141
6.1	Classes of Relations	141
6.1.1	Totality	141
6.1.2	Surjectivity	143
6.1.3	Injectivity	144
6.1.4	Functionality	146
6.1.5	Combining the Results	148
6.2	Relating a Type to Itself	149
6.2.1	Properties of These Relations	149
6.2.2	Order-Inducing Relations	152
6.2.3	Equivalence Relations	153
6.3	Summary	154
7	More Discrete Structures	157
7.1	Multisets	157
7.2	Sequences	160
7.3	Graph Theory	164
8	Defining New Structured Types	175
8.1	Notions and Notation	176
8.1.1	Simple Enumerated Types	176
8.1.2	More Elaborate Types	177
8.1.3	Self-Referential Types	179
8.1.4	Parametrised Types	181
8.2	Reasoning About New Types	184
8.2.1	Axiomatising New Types	184
8.2.2	A General Inductive Principle	188
8.2.3	Structural Induction	193
8.3	Using Structured Types	194
8.3.1	Three-Valued Logic	194
8.3.2	Processing Data	198
8.3.3	Lists	202
8.3.4	Binary Trees	206
8.4	Summary	210

- 9 Numbers 211**
 - 9.1 Natural Numbers 211
 - 9.1.1 Defining the Counting Numbers 212
 - 9.1.2 Defining the Arithmetic Operators 212
 - 9.1.3 Strong Induction 222
 - 9.1.4 Division and Prime Numbers 224
 - 9.2 Beyond the Natural 234
 - 9.2.1 Integers 235
 - 9.2.2 The Rational Numbers 237
 - 9.2.3 The Real Numbers 237
 - 9.3 Cardinality 239
 - 9.3.1 Counting with Finite Sets 239
 - 9.3.2 Extending Cardinality to Infinite Sets 249
 - 9.4 Summary 255
- 10 Reasoning About Programs 257**
 - 10.1 Correctness of Algorithms 258
 - 10.1.1 Euclid’s Algorithm 259
 - 10.1.2 Sorted Binary Trees 263
 - 10.2 Assigning Meaning to Programs 268
 - 10.2.1 Numeric Expressions 268
 - 10.2.2 Program Semantics 275
 - 10.3 The Uncomputable 280
 - 10.3.1 Counting Computer Programs 280
 - 10.3.2 Sets of Numbers 281
 - 10.3.3 Parsing Languages 282
 - 10.3.4 The Halting Problem 283
 - 10.4 Summary 285
- Index 287**