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\* Sections marked by an asterisk are recommended for a first course.  
Of sections marked <sup>(\*)</sup>, the beginning is recommended for a first course.

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