

Contents

1	The First Problem of Algebraic Regression	
	$\{\mathbf{Ax} = \mathbf{y} \mathbf{A} \in \mathbb{R}^{n \times m}, \mathbf{y} \in \mathcal{R}(\mathbf{A}) \sim \text{rk } \mathbf{A} = n = \dim \mathbb{Y}\}$, MINOS	1
1-1	Introduction	4
1-11	The Front Page Example	5
1-12	The Front Page Example: Matrix Algebra	5
1-13	The Front Page Example: MINOS, Horizontal Rank Partitioning	8
1-14	The Range $\mathcal{R}(f)$ and the Kernel $\mathcal{N}(f)$	10
1-15	The Interpretation of MINOS	12
1-2	Minimum Norm Solution (MINOS)	16
1-21	A Discussion of the Metric of the Parameter Space \mathbb{X}	21
1-22	An Alternative Choice of the Metric of the Parameter Space \mathbb{X}	22
1-23	\mathbf{G}_X -MINOS and Its Generalized Inverse	22
1-24	Eigenvalue Decomposition of \mathbf{G}_X -MINOS: Canonical MINOS	24
1-3	Case Study	37
1-31	Fourier Series	38
1-32	Fourier–Legendre Series	49
1-33	Nyquist Frequency for Spherical Data	62
1-4	Special Nonlinear Models	63
1-41	Taylor Polynomials, Generalized Newton Iteration	63
1-42	Linearized Models with Datum Defect	69
1-5	Notes	78

2	The First Problem of Probabilistic Regression:	
	The Bias Problem	
	Special Gauss–Markov Model with Datum Defects, LUMBE	81
2-1	Linear Uniformly Minimum Bias Estimator (LUMBE)	84
2-2	The Equivalence Theorem of \mathbf{G}_X -MINOS and \mathbf{S} -LUMBE	87
2-3	Example	88
3	The Second Problem of Algebraic Regression	
	Inconsistent system of linear observational equations	89
3-1	Introduction	92
3-11	The Front Page Example	92
3-12	The Front Page Example in Matrix Algebra	93
3-13	Least Squares Solution of the Front Page Example by Means of Vertical Rank Partitioning	95
3-14	The Range $\mathcal{R}(f)$ and the Kernel $\mathcal{N}(f)$, Interpretation of “LESS” by Three Partitionings	98
3-2	The Least Squares Solution: “LESS”	105
3-21	A Discussion of the Metric of the Parameter Space \mathbb{X}	112
3-22	Alternative Choices of the Metric of the Observation \mathbb{Y}	113
3-23	\mathbf{G}_X -LESS and Its Generalized Inverse	129
3-24	Eigenvalue Decomposition of \mathbf{G}_Y -LESS: Canonical LESS	130
3-3	Case Study	141
3-31	Canonical Analysis of the Hat Matrix, Partial Redundancies, High Leverage Points	142
3-32	Multilinear Algebra, “Join” and “Meet”, the Hodge Star Operator	150
3-33	From A to B: Latent Restrictions, Grassmann Coordinates, Plücker Coordinates	156
3-34	From B to A: Latent Parametric Equations, Dual Grassmann Coordinates, Dual Plücker Coordinates	168
3-35	Break Points	172
3-4	Special Linear and Nonlinear Models: A Family of Means for Direct Observations	180
3-5	A Historical Note on C.F. Gauss and A.M. Legendre	180
4	The Second Problem of Probabilistic Regression	
	Special Gauss-Markov model without datum defect	183
4-1	Introduction	187
4-11	The Front Page Example	188
4-12	Estimators of Type BLUE and BIQUUE of the Front Page Example	189

4-13	BLUE and BIQUUE of the Front Page Example, Sample Median, Median Absolute Deviation	198
4-14	Alternative Estimation Maximum Likelihood (MALE)	202
4-2	Setup of the Best Linear Uniformly Unbiased Estimator	205
4-21	The Best Linear Uniformly Unbiased Estimation $\hat{\xi}$ of $\xi : \Sigma_y$ -BLUE	206
4-22	The Equivalence Theorem of G_y -LESS and Σ_y -BLUE	213
4-3	Setup of the Best Invariant Quadratic Uniformly Unbiased Estimator	214
4-31	Block Partitioning of the Dispersion Matrix and Linear Space Generated by Variance-Covariance Components	215
4-32	Invariant Quadratic Estimation of Variance-Covariance Components of Type IQE	220
4-33	Invariant Quadratic Uniformly Unbiased Estimations of Variance-Covariance Components of Type IQUE	224
4-34	Invariant Quadratic Uniformly Unbiased Estimations of One Variance Component (IQUE) from Σ_y -BLUE: HIQUUE	228
4-35	Invariant Quadratic Uniformly Unbiased Estimators of Variance Covariance Components of Helmert Type: HIQUUE Versus HIQE	230
4-36	Best Quadratic Uniformly Unbiased Estimations of One Variance Component: BIQUUE....	234
4-37	Simultaneous Determination of First Moment and the Second Central Moment, Inhomogeneous Multilinear Estimation, the $E - D$ Correspondence, Bayes Design with Moment Estimations	241
5	The Third Problem of Algebraic Regression	
	$\{Ax + i = y A \in \mathbb{R}^{n \times m}, y \notin \mathcal{R}(A) \sim \text{rk} A < \min\{m, n\}\}$	263
5-1	Introduction	265
5-11	The Front Page Example	266
5-12	The Front Page Example in Matrix Algebra	266
5-13	Minimum Norm: Least Squares Solution of the Front Page Example by Means of Additive Rank Partitioning	268

5-14	Minimum Norm: Least Squares Solution of the Front Page Example by Means of Multiplicative Rank Partitioning	272
5-15	The Range $R(f)$ and the Kernel $N(f)$ Interpretation of "MINOLESS" by Three Partitionings	276
5-2	MINOLESS and Related Solutions Like Weighted Minimum Norm-Weighted Least Squares Solutions	283
5-21	The Minimum Norm-Least Squares Solution: "MINOLESS"	283
5-22	(G_x, G_y) -MINOS and Its Generalized Inverse	293
5-23	Eigenvalue Decomposition of (G_x, G_y) -MINOLESS	297
5-24	Notes	301
5-3	The Hybrid Approximation Solution: α -HAPS and Tykhonov-Phillips Regularization	302
6	The Third Problem of Probabilistic Regression	
	Special Gauss-Markov model with datum defect	305
6-1	Setup of the Best Linear Minimum Bias Estimator of Type BLUMBE	308
6-11	Definitions, Lemmas and Theorems	310
6-12	The First Example: BLUMBE Versus BLE, BIQUUE Versus BIQE, Triangular Leveling Network	317
6-2	Setup of the Best Linear Estimators of Type hom BLE, hom S-BLE and hom a-BLE for Fixed Effects	332
6-3	Continuous Networks	345
6-31	Continuous Networks of Second Derivatives Type	346
6-32	Discrete Versus Continuous Geodetic Networks	357
7	Overdetermined System of Nonlinear Equations on Curved Manifolds	
	inconsistent system of directional observational equations	361
7-1	Introduction	362
7-2	Minimal Geodesic Distance: MINGEODISC	365
7-3	Special Models: From the Circular Normal Distribution to the Oblique Normal Distribution	370
7-31	A Historical Note of the von Mises Distribution	370
7-32	Oblique Map Projection	372
7-33	A Note on the Angular Metric	375
7-4	Case Study	376

8	The Fourth Problem of Probabilistic Regression	
	Special Gauss–Markov model with random effects	383
8-1	The Random Effect Model	384
8-2	Examples	399
9	The Fifth Problem of Algebraic Regression: The System of Conditional Equations: Homogeneous and Inhomogeneous Equations: $\{By = Bi \text{ versus } -c + By = Bi\}$	411
9-1	G_y -LESS of a System of a Inconsistent Homogeneous Conditional Equations	411
9-2	Solving a System of Inconsistent Inhomogeneous Conditional Equations	415
9-3	Examples	416
10	The Fifth Problem of Probabilistic Regression	
	general Gauss–Markov model with mixed effects	419
10-1	Inhomogeneous General Linear Gauss–Markov Model Fixed Effects and Random Effects	421
10-2	Explicit Representations of Errors in the General Gauss–Markov Model with Mixed Effects	426
10-3	An Example for Collocation	428
10-4	Comments	438
11	The Sixth Problem of Probabilistic Regression	
	- The Random Effect Model – “errors-in-variable”	443
11-1	The Model of Error-in-Variables or Total Least Squares	447
11-2	Algebraic Total Least Squares for the Nonlinear System of the Model “Error-in-Variables”	448
11-3	Example: The Straight Line Fit	450
11-4	The Models SIMEX and SYMEX	453
11-5	References	459
12	The Nonlinear Problem of the 3d Datum Transformation and the Procrustes Algorithm	461
12-1	The 3d Datum Transformation and the Procrustes Algorithm ...	463
12-2	The Variance: Covariance Matrix of the Error Matrix E	470
	12-21 Case Studies: The 3d Datum Transformation and the Procrustes Algorithm	471
12-3	References	474
13	The Sixth Problem of Generalized Algebraic Regression	
	the system of conditional equations with unknowns - (Gauss–Helmert model)	477
13-1	Variance-Covariance-Component Estimation in the Linear Model $Ax + \varepsilon = y, y \notin \mathcal{R}(A)$	479
13-2	Variance-Covariance-Component Estimation in the Linear Model $B\varepsilon = By - c, By \notin \mathcal{R}(A) + c$	482

13-3	Variance-Covariance-Component Estimation in the Linear Model $\mathbf{Ax} + \varepsilon + \mathbf{B}\varepsilon = \mathbf{By} - \mathbf{c}$, $\mathbf{By} \notin \mathcal{R}(\mathbf{A}) + \mathbf{c}$	485
13-4	The Block Structure of Dispersion Matrix $\mathbf{D}\{\mathbf{y}\}$	489
14	Special Problems of Algebraic Regression and Stochastic Estimation	493
14-1	The Multivariate Gauss–Markov Model: A Special Problem of Probabilistic Regression	493
14-2	n -Way Classification Models	498
14-21	A First Example: 1-Way Classification	499
14-22	A Second Example: 2-Way Classification Without Interaction	503
14-23	A Third Example: 2-Way Classification with Interaction	509
14-24	Higher Classifications with Interaction	514
14-3	Dynamical Systems	517
15	Algebraic Solutions of Systems of Equations Linear and Nonlinear Systems of Equations	527
15-1	Introductory Remarks	527
15-2	Background to Algebraic Solutions	528
15-3	Algebraic Methods for Solving Nonlinear Systems of Equations	532
15-31	Solution of Nonlinear Gauss–Markov Model	532
15-32	Adjustment of the combinatorial subsets	552
15-4	Examples	556
15-5	Notes	563
A	Tensor Algebra, Linear Algebra, Matrix Algebra, Multilinear Algebra	571
A-1	Multilinear Functions and the Tensor Space \mathbb{T}_q^p	572
A-2	Decomposition of Multilinear Functions into Symmetric Multilinear Functions Antisymmetric Multi-linear Functions and Residual Multilinear Functions $\mathbb{T}T_q^p = \mathbb{S}_q^p \oplus \mathbb{A}_q^p \oplus \mathbb{R}_q^p$	578
A-3	Matrix Algebra, Array Algebra, Matrix Norm and Inner Product	584
A-4	The Hodge Star Operator, Self Duality	587
A-5	Linear Algebra	592
A-51	Definition of a Linear Algebra	593
A-52	The Diagrams “Ass”, “Uni” and “Comm”	595
A-53	Ringed Spaces: The Subalgebra “Ring with Identity”	597
A-54	Definition of a Division Algebra and Non-Associative Algebra	598

A-55	Lie Algebra, Witt Algebra	598
A-56	Definition of a Composition Algebra	599
A-6	Matrix Algebra Revisited, Generalized Inverses	602
A-61	Special Matrices: Helmholtz Matrix, Hankel Matrix, Vandemonte Matrix	606
A-62	Scalar Measures of Matrices	612
A-63	Three Basic Types of Generalized Inverses.....	618
A-7	Complex Algebra, Quaternion Algebra, Octonian Algebra, Clifford Algebra, Hurwitz Theorem	619
A-71	Complex Algebra as a Division Algebra as well as a Composition Algebra, <i>Clifford algebra</i> $Cl(0, 1)$	620
A-72	Quaternion Algebra as a Division Algebra as well as a Composition Algebra, <i>Clifford algebra</i> $Cl(0, 2)$	622
A-73	Octonian Algebra as a Non-Associative Algebra as well as a Composition Algebra, <i>Clifford algebra</i> with Respect to $\mathbb{H} \times \mathbb{H}$	629
A-74	Clifford Algebra	633
B	Sampling Distributions and Their Use: Confidence Intervals and Confidence Regions	637
B-1	A First Vehicle: Transformation of Random Variables.....	638
B-2	A Second Vehicle: Transformation of Random Variables.....	642
B-3	A First Confidence Interval of Gauss–Laplace Normally Distributed Observations μ, σ^2 Known, the Three Sigma Rule	648
B-31	The Forward Computation of a First Confidence Interval of Gauss–Laplace Normally Distributed Observations: μ, σ^2 Known	653
B-32	The Backward Computation of a First Confidence Interval of Gauss–Laplace Normally Distributed Observations: μ, σ^2 Known	659
B-4	Sampling from the Gauss– Laplace Normal Distribution: A Second Confidence Interval for the Mean, Variance Known	662
B-41	Sampling Distributions of the Sample Mean $\hat{\mu}, \sigma^2$ Known, and of the Sample Variance $\hat{\sigma}^2$	677
B-42	The Confidence Interval for the Sample Mean, Variance Known	688
B-5	Sampling from the Gauss– Laplace Normal Distribution: A Third Confidence Interval for the Mean, Variance Unknown	692
B-51	Student’s Sampling Distribution of the Random Variable $(\hat{\mu} - \mu)/\hat{\sigma}$	692

B-52	The Confidence Interval for the Mean, Variance Unknown	701
B-53	The Uncertainty Principle	707
B-6	Sampling from the Gauss– Laplace Normal Distribution: A Fourth Confidence Interval for the Variance	708
B-61	The Confidence Interval for the Variance	709
B-62	The Uncertainty Principle	715
B-7	Sampling from the Multidimensional Gauss–Laplace Normal Distribution: The Confidence Region for the Fixed Parameters in the Linear Gauss–Markov Model	717
B-8	Multidimensional Variance Analysis, Sampling from the Multivariate Gauss–Laplace Normal Distribution	739
B-81	Distribution of Sample Mean and Variance-Covariance	740
B-82	Distribution Related to Correlation Coefficients	744
C	Statistical Notions, Random Events and Stochastic Processes	753
C-1	Moments of a Probability Distribution, the Gauss– Laplace Normal Distribution and the Quasi-Normal Distribution	754
C-2	Error Propagation	757
C-3	Useful Identities	760
C-4	Scalar – Valued Stochastic Processes of One Parameter	762
C-5	Characteristic of One Parameter Stochastic Processes	765
C-6	Simple Examples of One Parameter Stochastic Processes	769
C-7	Wiener Processes	781
C-71	Definition of the Wiener Processes	781
C-72	Special Wiener Processes: Ornstein– Uhlenbeck, Wiener Processes with Drift, Integral Wiener Processes	785
C-8	Special Analysis of One Parameter Stationary Stochastic Process	793
C-81	Foundations: Ergodic and Stationary Processes	793
C-82	Processes with Discrete Spectrum	795
C-83	Processes with Continuous Spectrum	798
C-84	Spectral Decomposition of the Mean and Variance-Covariance Function	808
C-9	Scalar-, Vector-, and Tensor Valued Stochastic Processes of Multi-Parameter Systems	811
C-91	Characteristic Functional	812
C-92	The Moment Representation of Stochastic Processes for Scalar Valued and Vector Valued Quantities	814
C-93	Tensor-Valued Statistical Homogeneous and Isotropic Field of Multi-Point Systems	818

D	Basics of Groebner Basis Algebra	887
D-1	Definitions	887
D-2	Buchberger Algorithm	889
D-21	Mathematica Computation of Groebner Basis	889
D-22	Maple Computation of Groebner Basis	891
D.3	Gauss Combinatorial Formulation	892
	References.....	895
	Index	1011