

Contents

Preface — V

1	Fractional derivatives and numerical approximations — 1
1.1	Definitions and properties of fractional derivatives — 1
1.1.1	Fractional integral — 1
1.1.2	Grünwald–Letnikov fractional derivative — 1
1.1.3	Riemann–Liouville fractional derivative — 2
1.1.4	Caputo fractional derivative — 3
1.1.5	Riesz fractional derivative — 4
1.1.6	Behaviors of fractional derivatives at the lower limit of integrals — 5
1.2	The Fourier transform of fractional derivatives — 5
1.3	Fractional ordinary differential equations — 7
1.3.1	Solutions to the FODEs in Riemann–Liouville type — 7
1.3.2	Solutions to the FODEs in Caputo type — 10
1.4	G–L approximations of Riemann–Liouville fractional derivatives — 11
1.5	Central difference quotient approximations of Riesz fractional derivatives — 25
1.6	Interpolation approximations of Caputo fractional derivatives — 31
1.6.1	L1 approximation — 31
1.6.2	L1–2 approximation — 39
1.6.3	L2–1 _{σ} approximation — 42
1.6.4	L2–1 _{σ} approximation of multi-term fractional derivatives — 49
1.6.5	H2N2 approximation — 57
1.7	Fast interpolation approximations of Caputo fractional derivatives — 67
1.7.1	Fast L1 approximation — 67
1.7.2	Fast L2–1 _{σ} approximation — 74
1.7.3	Fast H2N2 approximation — 81
1.8	Finite difference methods for FODEs — 85
1.8.1	Method based on G–L approximation — 85
1.8.2	Method based on L1 approximation — 93
1.8.3	Method based on L2–1 _{σ} approximation — 98
1.9	A simple classification of the fractional partial differential equations — 101
1.10	Supplementary remarks and discussions — 103
	Exercises 1 — 106
2	Difference methods for the time-fractional subdiffusion equations — 109
2.1	The second-order method in space based on G–L approximation for 1D problem — 109

2.1.1	Derivation of the difference scheme —	111
2.1.2	Solvability of the difference scheme —	112
2.1.3	Stability of the difference scheme —	113
2.1.4	Convergence of the difference scheme —	115
2.2	The fourth-order method in space based on G-L approximation for 1D problem —	116
2.2.1	Derivation of the difference scheme —	116
2.2.2	Solvability of the difference scheme —	117
2.2.3	Stability of the difference scheme —	117
2.2.4	Convergence of the difference scheme —	118
2.3	The second-order method in space based on L1 approximation for 1D problem —	119
2.3.1	Derivation of the difference scheme —	119
2.3.2	Solvability of the difference scheme —	120
2.3.3	Stability of the difference scheme —	121
2.3.4	Convergence of the difference scheme —	122
2.4	The fast difference method based on L1 approximation for 1D problem —	123
2.4.1	Derivation of the difference scheme —	123
2.4.2	Solvability of the difference scheme —	125
2.4.3	Stability of the difference scheme —	126
2.4.4	Convergence of the difference scheme —	127
2.5	The fourth-order method in space based on L1 approximation for 1D problem —	128
2.5.1	Derivation of the difference scheme —	128
2.5.2	Solvability of the difference scheme —	129
2.5.3	Stability of the difference scheme —	130
2.5.4	Convergence of the difference scheme —	131
2.6	The difference method based on $L2-1_\sigma$ approximation for 1D problem —	132
2.6.1	Derivation of the difference scheme —	132
2.6.2	Solvability of the difference scheme —	133
2.6.3	An important lemma —	134
2.6.4	Stability of the difference scheme —	136
2.6.5	Convergence of the difference scheme —	139
2.7	The fast difference method based on $L2-1_\sigma$ approximation for 1D problem —	140
2.7.1	Derivation of the difference scheme —	140
2.7.2	Solvability of the difference scheme —	141
2.7.3	Stability of the difference scheme —	142
2.7.4	Convergence of the difference scheme —	144

2.8	The difference method based on L1 approximation for the MTTFSD equations — 145
2.8.1	Derivation of the difference scheme — 146
2.8.2	Solvability of the difference scheme — 147
2.8.3	Stability of the difference scheme — 147
2.8.4	Convergence of the difference scheme — 149
2.9	The difference method based on L2-1 _σ approximation for the MTTFSD equations — 150
2.9.1	Derivation of the difference scheme — 151
2.9.2	Solvability of the difference scheme — 152
2.9.3	Stability of the difference scheme — 152
2.9.4	Convergence of the difference scheme — 154
2.10	The ADI method based on G-L approximation for 2D problem — 155
2.10.1	Derivation of the difference scheme — 156
2.10.2	Solvability of the difference scheme — 158
2.10.3	Stability of the difference scheme — 159
2.10.4	Convergence of the difference scheme — 161
2.11	The ADI method based on L1 approximation for 2D problem — 162
2.11.1	Derivation of the difference scheme — 162
2.11.2	Solvability of the difference scheme — 164
2.11.3	Stability of the difference scheme — 165
2.11.4	Convergence of the difference scheme — 166
2.12	Supplementary remarks and discussions — 167
	Exercises 2 — 170
3	Difference methods for the time-fractional wave equations — 173
3.1	The second-order method in space based on L1 approximation for 1D problem — 173
3.1.1	Derivation of the difference scheme — 173
3.1.2	Solvability of the difference scheme — 174
3.1.3	Stability of the difference scheme — 175
3.1.4	Convergence of the difference scheme — 178
3.2	The fast difference method based on L1 approximation for 1D problem — 179
3.2.1	Derivation of the difference scheme — 179
3.2.2	Solvability of the difference scheme — 180
3.2.3	Stability of the difference scheme — 181
3.2.4	Convergence of the difference scheme — 186
3.3	The fourth-order method in space based on L1 approximation for 1D problem — 187
3.3.1	Derivation of the difference scheme — 187
3.3.2	Solvability of the difference scheme — 188

3.3.3	Stability of the difference scheme —	189
3.3.4	Convergence of the difference scheme —	191
3.4	The difference method based on $L_2\text{-}1_\sigma$ approximation for 1D problem —	192
3.4.1	Derivation of the difference scheme —	193
3.4.2	Solvability of the difference scheme —	196
3.4.3	Stability of the difference scheme —	198
3.4.4	Convergence of the difference scheme —	208
3.5	The fast difference method based on $L_2\text{-}1_\sigma$ approximation for 1D problem —	209
3.5.1	Derivation of the difference scheme —	210
3.5.2	Solvability of the difference scheme —	212
3.5.3	Stability of the difference scheme —	214
3.5.4	Convergence of the difference scheme —	221
3.6	The difference method based on L_1 approximation for the MTTFW equations —	222
3.6.1	Derivation of the difference scheme —	222
3.6.2	Solvability of the difference scheme —	223
3.6.3	Stability of difference scheme —	224
3.6.4	Convergence of the difference scheme —	226
3.7	The difference method based on $L_2\text{-}1_\sigma$ approximation for the MTTFW equations —	227
3.7.1	Derivation of the difference scheme —	227
3.7.2	Solvability of the difference scheme —	230
3.7.3	Stability of the difference scheme —	232
3.7.4	Convergence of the difference scheme —	239
3.8	The difference method for the time-fractional mixed diffusion and wave equations —	240
3.8.1	Derivation of the difference scheme —	240
3.8.2	Solvability of the difference scheme —	242
3.8.3	Stability of the difference scheme —	242
3.8.4	Convergence of the difference scheme —	245
3.9	The ADI method based on L_1 approximation for 2D problem —	246
3.9.1	Derivation of the difference scheme —	247
3.9.2	Solvability of the difference scheme —	249
3.9.3	Stability of the difference scheme —	250
3.9.4	Convergence of the difference scheme —	252
3.10	The compact ADI method based on L_1 approximation for 2D problem —	253
3.10.1	Derivation of the difference scheme —	253
3.10.2	Solvability of the difference scheme —	256
3.10.3	Stability of the difference scheme —	257

3.10.4	Convergence of the difference scheme —	260
3.11	Supplementary remarks and discussions —	261
	Exercises 3 —	263
4	Difference methods for the space-fractional partial differential equations —	269
4.1	The first-order method based on the shifted G-L formula for 1D problem —	269
4.1.1	Derivation of the difference scheme —	270
4.1.2	Solvability of the difference scheme —	271
4.1.3	Stability of the difference scheme —	272
4.1.4	Convergence of the difference scheme —	273
4.2	The second-order method based on the WSGL formula for 1D problem —	273
4.2.1	Derivation of the difference scheme —	274
4.2.2	Solvability of the difference scheme —	275
4.2.3	Stability of the difference scheme —	276
4.2.4	Convergence of the difference scheme —	277
4.3	The fourth-order method based on the WSGL formula for 1D problem —	278
4.3.1	Derivation of the difference scheme —	279
4.3.2	Solvability of the difference scheme —	280
4.3.3	Stability of the difference scheme —	281
4.3.4	Convergence of the difference scheme —	283
4.4	The fourth-order ADI method based on the WSGL formula for 2D problem —	284
4.4.1	Derivation of the difference scheme —	285
4.4.2	Three lemmas —	287
4.4.3	Solvability of the difference scheme —	289
4.4.4	Stability of the difference scheme —	290
4.4.5	Convergence of the difference scheme —	292
4.5	Supplementary remarks and discussions —	293
	Exercises 4 —	294
5	Difference methods for the time-space-fractional differential equations —	297
5.1	The second-order method in space for 1D problem —	297
5.1.1	Derivation of the difference scheme —	298
5.1.2	Solvability of the difference scheme —	299
5.1.3	An important lemma —	300
5.1.4	Stability of the difference scheme —	302
5.1.5	Convergence of the difference scheme —	304

5.2	The fourth-order method in space for 1D problem —	305
5.2.1	Derivation of the difference scheme —	305
5.2.2	Solvability of the difference scheme —	307
5.2.3	Stability of the difference scheme —	308
5.2.4	Convergence of the difference scheme —	309
5.3	The second-order method in space for 2D problem —	310
5.3.1	Derivation of the difference scheme —	311
5.3.2	Solvability of the difference scheme —	312
5.3.3	Stability of the difference scheme —	313
5.3.4	Convergence of the difference scheme —	315
5.4	The fourth-order method in space for 2D problem —	316
5.4.1	Derivation of the difference scheme —	317
5.4.2	Solvability of the difference scheme —	318
5.4.3	Stability of the difference scheme —	320
5.4.4	Convergence of the difference scheme —	323
5.5	Supplementary remarks and discussions —	323
	Exercises 5 —	325
6	Difference methods for the time distributed-order subdiffusion equations —	327
6.1	The second-order method in both space and distributed order for 1D problem —	327
6.1.1	Derivation of the difference scheme —	328
6.1.2	Solvability of the difference scheme —	330
6.1.3	Three lemmas —	330
6.1.4	Stability of the difference scheme —	333
6.1.5	Convergence of the difference scheme —	335
6.2	The fourth-order method in both space and distributed order for 1D problem —	336
6.2.1	Derivation of the difference scheme —	336
6.2.2	Solvability of the difference scheme —	338
6.2.3	Stability of the difference scheme —	339
6.2.4	Convergence of the difference scheme —	340
6.3	The second-order method in both space and distributed order for 2D problem —	342
6.3.1	Derivation of the difference scheme —	343
6.3.2	Solvability of the difference scheme —	344
6.3.3	Stability of the difference scheme —	345
6.3.4	Convergence of the difference scheme —	346
6.4	The fourth-order method in both space and distributed order for 2D problem —	347
6.4.1	Derivation of the difference scheme —	347

6.4.2	Solvability of the difference scheme —	349
6.4.3	Stability of the difference scheme —	349
6.4.4	Convergence of the difference scheme —	351
6.5	The second-order ADI method in both space and distributed-order for 2D problem —	352
6.5.1	Derivation of the difference scheme —	352
6.5.2	Solvability of the difference scheme —	354
6.5.3	Stability of the difference scheme —	355
6.5.4	Convergence of the difference scheme —	356
6.6	The fourth-order ADI method in both space and distributed-order for 2D problem —	357
6.6.1	Derivation of the difference scheme —	358
6.6.2	Solvability of the difference scheme —	359
6.6.3	Stability of the difference scheme —	360
6.6.4	Convergence of the difference scheme —	361
6.7	Supplementary remarks and discussions —	362
	Exercises 6 —	365

A	The Matlab code of sum-of-exponentials approximations for the kernel $t^{-\alpha}$ in the Caputo fractional derivative —	369
----------	--	------------

Bibliography —	373
-----------------------	------------

Index —	379
----------------	------------