

# Contents

## Basic notation — XIII

Sergey Lemeshevsky, Piotr Matus, and Dmitriy Poliakov

## Preliminary results — 1

1 Exact finite-difference schemes for ordinary differential equations — 1

2 Effective computations of integrals — 4

Bibliography — 6

Sergey Lemeshevsky, Piotr Matus, and Dmitriy Poliakov

## Hyperbolic equations — 7

1 Linear and semi-linear transport equation — 7

1.1 One-dimensional transport equation — 7

1.2 Two-dimensional transport equation — 14

1.3 Three-dimensional transport equation — 15

2 Exact numerical simulation of shock waves with a variable amplitude — 24

2.1 Two-dimensional advection-reaction equation — 27

3 Linear system of two equations — 28

3.1 Cauchy problem — 28

3.2 IBVP problem in Riemann invariants — 33

4 Riemann problem for non-linear transport equation — 44

4.1 Numerical experiment — 47

5 Non-homogeneous quasilinear transport equation — 49

5.1 Statement of the problem — 49

5.2 Finite-difference scheme — 50

5.3 Approximation error — 50

5.4 Stability of the finite-difference scheme — 52

5.5 Convergence — 56

5.6 Numerical experiment — 57

6 Linear equation of vibrations of a string — 62

6.1 Dirichlet IBVP — 62

6.2 Robin IBVP — 67

6.3 Numerical experiment for discontinuous input data — 72

7 Non-linear equation of vibrations of a string — 80

7.1 Statement of the problem — 80

7.2 Finite-difference schemes on characteristic grids — 81

7.3 Numerical experiment — 84

|            |                                           |
|------------|-------------------------------------------|
| <b>8</b>   | <b>Non-linear gas dynamic system — 86</b> |
| <b>8.1</b> | <b>Statement of the problem — 87</b>      |
| <b>8.2</b> | <b>Finite-difference scheme — 89</b>      |
| <b>8.3</b> | <b>Numerical experiment — 91</b>          |
|            | <b>Bibliography — 93</b>                  |

**Sergey Lemeshevsky, Piotr Matus, and Dmitriy Poliakov**

**Parabolic equations — 95**

|            |                                                                                            |
|------------|--------------------------------------------------------------------------------------------|
| <b>1</b>   | <b>Travelling wave type solutions — 95</b>                                                 |
| <b>1.1</b> | <b>One-dimensional parabolic equation — 95</b>                                             |
| <b>1.2</b> | <b>Two-dimensional convection-diffusion-reaction equation — 102</b>                        |
| <b>2</b>   | <b>Solutions of the separated variables — 115</b>                                          |
| <b>2.1</b> | <b>Introduction — 116</b>                                                                  |
| <b>2.2</b> | <b>The Cauchy problem for parabolic equations — 116</b>                                    |
| <b>2.3</b> | <b>Arbitrary-order finite-difference schemes for the BVP for parabolic equations — 122</b> |
| <b>2.4</b> | <b>The BVP for parabolic equations with small parameter — 126</b>                          |
| <b>2.5</b> | <b>The Cauchy problem for multidimensional parabolic equations — 129</b>                   |
| <b>3</b>   | <b><math>L_1</math>-conservative FDS for the Neumann problem — 133</b>                     |
| <b>3.1</b> | <b>Introduction — 133</b>                                                                  |
| <b>3.2</b> | <b>Exact <math>L_1</math>-conservative finite-difference scheme — 135</b>                  |
| <b>3.3</b> | <b>Exact conservative iteration process — 137</b>                                          |
| <b>3.4</b> | <b>Flow finite-difference scheme — 138</b>                                                 |
| <b>3.5</b> | <b>Exact <math>L_1</math>-conservative flow algorithm — 139</b>                            |
| <b>3.6</b> | <b>Multidimensional generalization — 140</b>                                               |
|            | <b>Bibliography — 142</b>                                                                  |

**Ronald E. Mickens and Talitha M. Washington**

**Use of exact difference schemes to construct NSFD discretizations of differential equations — 144**

|          |                                                            |
|----------|------------------------------------------------------------|
| <b>1</b> | <b>Introduction — 144</b>                                  |
| <b>2</b> | <b>Exact finite-difference schemes — 145</b>               |
| <b>3</b> | <b>Logistic equation — 146</b>                             |
| <b>4</b> | <b>Second-order ODE having constant coefficients — 146</b> |
| <b>5</b> | <b>Jacobi differential equations — 148</b>                 |
| <b>6</b> | <b>A non-linear reaction-advection PDE — 149</b>           |
| <b>7</b> | <b>Comment — 150</b>                                       |
| <b>8</b> | <b>NSFD methodology — 150</b>                              |
| <b>9</b> | <b>Discrete derivatives — 151</b>                          |

|    |                                             |     |
|----|---------------------------------------------|-----|
| 10 | Non-locality of functional terms —          | 152 |
| 11 | Other properties —                          | 152 |
| 12 | Dynamic consistency —                       | 153 |
| 13 | Subequations —                              | 153 |
| 14 | NSFD applications —                         | 154 |
| 15 | Conservative oscillators —                  | 154 |
| 16 | Time-independent Schrödinger equation —     | 155 |
| 17 | Linear, advective-diffusive PDE —           | 156 |
| 18 | Linear advection, non-linear reaction PDE — | 158 |
| 19 | Combustion model —                          | 159 |
| 20 | Coupled interacting population —            | 160 |
| 21 | Black-Scholes equation —                    | 162 |
| 22 | Summary —                                   | 163 |
|    | Bibliography —                              | 163 |

Ivan Gavrilyuk, Myroslav Kutniv, and Volodymyr Makarov

**Exact and truncated difference schemes for boundary-value problem — 165**

|   |                                                                                            |     |
|---|--------------------------------------------------------------------------------------------|-----|
| 1 | Introduction —                                                                             | 165 |
| 2 | Three-point difference schemes of high-order accuracy for linear boundary-value problems — | 168 |
| 3 | Exact and truncated difference schemes for non-linear boundary-value problems —            | 174 |
| 4 | Two-point difference schemes for systems of first-order ODEs —                             | 175 |
| 5 | Three-point difference schemes for non-linear second-order ODEs —                          | 178 |
| 6 | Three-point difference schemes for non-linear BVPs on the half-axis —                      | 184 |
| 7 | Three-point difference schemes for singular non-linear BVPs —                              | 191 |
| 8 | Exact- difference schemes for PDEs —                                                       | 196 |
|   | Bibliography —                                                                             | 200 |

Silvia Jerez and Saúl Díaz-Infante

**Exact difference schemes for stochastic differential equations — 204**

|   |                             |     |
|---|-----------------------------|-----|
| 1 | Introduction —              | 204 |
| 2 | General settings —          | 205 |
| 3 | Steklov method —            | 206 |
| 4 | Convergence and Stability — | 209 |
| 5 | Linear Steklov method —     | 210 |
| 6 | Almost sure stability —     | 213 |
| 7 | Numerical simulations —     | 215 |
|   | Bibliography —              | 218 |

**Ryszard Kozera, Agnieszka Paradzinska, and Denis Schadinskii**

**Numerical blow-up time — 220**

- 1 Introduction — 220**
- 2 The Cauchy problem for ODE — 221**
- 3 Statement of the problem and FDS — 221**
- 4 Solvability of the implicit FDS for ODE — 225**
- 5 Numerical experiment — 226**
- 6 The Neumann problem for a parabolic equation with a non-linear source  
of power form — 228**

**Bibliography — 231**

**List of contributors — 233**