Petr Hájek Pavel Pudlák

Metamathematics of First-Order Arithmetic



Table of Contents

Introduction	٠	•	•	•	٠	•	1
Preliminaries (a) Some Logic (b) The Language of Arithmetic, the Standard Model (c) Beginning Arithmetization of Metamathematics	•						12
PART A							
CHAPTER I							
Arithmetic as Number Theory, Set Theory and Logic .							27
Introduction							27
1. Basic Developments; Partial Truth Definitions							28
(a) Properties of Addition and Multiplication,							
Divisibility and Primes							28
(b) Coding Finite Sets and Sequences; the Theory $I\Sigma$	տ(exp)				37
(c) Provably Recursive Functions; the Theory $I\Sigma_1$.		_	•				44
(d) Arithmetization of Metamathematics: Partial True							50
2. Fragments of First-Order Arithmetic							61
(a) Induction and Collection							61
(b) Further Principles and Facts About Fragments .							67
(c) Finite Axiomatizability; Partial Truth Definitions		-			-	-	,
for Relativized Arithmetical Formulas					_		77
(d) Relativized Hierarchy in Fragments							81
(e) Axiomatic Systems of Arithmetic with No Function							86
3. Fragments and Recursion Theory							89
(a) Limit Theorem							89
(b) Low Basis Theorem							91
(c) Infinite Δ_1 Subsets	•	•	•	•	•	•	95
(d) Matiyasevič's Theorem in $I\Sigma_1$							97
4. Elements of Logic in Fragments							98
(a) Arithmetizing Provability	•	•	•	•	•	•	98
(-)	•	•	•	•	•	•	-

XII Table of Contents

(b) Arithmetizing Model Theory
(c) Applications to Arithmetic
CHAPTER II
Fragments and Combinatorics
1. Ramsey's Theorems and Fragments
(a) Statement of Results
(b) Proofs (of 1.5, 1.7, 1.9)
(c) Proofs (of 1.6, 1.8, 1.10)
2. Instances of the Paris-Harrington Principle
and Consistency Statements
(a) Introduction and Statement of Results
(b) Some Combinatorics
(d) Strong Indiscernibles $\dots \dots \dots$
(e) Final Considerations
3. Schwichtenberg-Wainer Hierarchy and α-large Sets
(a) Ordinals in $I\Sigma_1$
(a) Ordinals in 121
• • • • • • • • • • • • • • • • • • • •
(d) Schwichtenberg-Wainer Hierarchy
PART B
CHAPTER III
Self-Reference
1. Preliminaries
(a) Interpretability and Partial Conservativity
(b) Theories Containing Arithmetic; Sequential Theories;
PA and ACA ₀
(c) Numerations and Binumerations
2. Self-Reference and Gödel's Theorems, Reflexive Theories
(a) Existence of Fixed Points
(b) Gödel's First Incompleteness Theorem and Related Topics 160
(c) Gödel's Second Incompleteness Theorem
(d) Pure Extensions of PA
(e) Interpretability in Pure Extensions of PA
3. Definable Cuts
(a) Definable Cuts and Their Properties
(b) A Strong Form of Gödel's Second Incompleteness Theorem . 173
(c) Herbrand Provability and Herbrand Consistency 179
(d) Cuts and Interpretations
4. Partial Conservativity and Interpretability
(a) Some Prominent Examples

	•	
	Table of Contents	XIII
(b) General Theorems on Partial Conservativity	r:	
Some Fixed-Point Theorems		195
(c) Applications, Mainly to Interpretability .		206
		200
CHAPTER IV		
Models of Fragments of Arithmetic		213
1. Some Basic Constructions		214
(a) Preliminaries		214
(b) Definable Ultrapower of the Standard Mode		216
(c) On Submodels and Cuts		218
		220
(e) Elementary End Extensions		227
(f) A Conservation Result		230
2. Cuts in Models of Arithmetic with a Top	• • • • • • • •	232
(a) Arithmetic with a Top and Its Models .		232
(b) Cuts		
(c) Extendable, Restrainable and Ramsey Cuts		236
(d) Satisfaction in Finite Structures with an Ap		041
to Models of $I\Sigma_1$		241
3. Provably Recursive Functions and the Method of		
(a) Provably Recursive Functions, Envelopes		
(b) Indicators and Paris Sequences		
(c) Paris Sequences of the First Kind		250
(d) Paris Sequences of the Second Kind		253
(e) Further Consequences		257
4. Formalizing Model Theory		258
(a) Some Results on Satisfaction and Consisten	су	259
(b) A Conservation Result in $I\Sigma_1$		260
(c) Appendix: Another Conservation Result .		
(s) soft and an analysis of the second secon		
PART C		
CHAPTER V	ı	
7		267
1. A Survey of Weak Fragments of Arithmetic .	• • • • • • • •	268
(a) Fragments of Arithmetic	• • • • • • • •	268
2. A Brief Introduction to Complexity Theory .	· · · · · · · · ·	276
(a) Time and Space Complexity Classes		277
(b) Nondeterministic Computations		279
(c) Degrees and NP-completeness		280
(d) Oracle Computations		282
(e) The Linear Time Hierarchy and the Polynon	mial Hierarchy	283
(f) Nepomnjaščij's Theorem		285
(g) The Diagonal Method for Separating Comp.	lexity Classes	288
	· · · · · · · · · · · · · · · · · · ·	

XIV Table of Contents

3.	Exponentiation, Coding Sequences				
	and Formalization of Syntax in $I\Sigma_0$				294
	(a) Introduction				294
	(b) Sets and Sequences				295
	(c) The Exponentiation Relation				299
	(d) Developing $I\Sigma_0 + \Omega_1$				303
	(e) The Number of Ones in a Binary Expansion				304
	(f) Coding Sequences				309
	(g) Syntactical Concepts				312
	(h) Formalizations Based on Context-Free Grammars .				315
4.	Witnessing Functions				320
	(a) Introduction				320
	(b) Fragments of Bounded Arithmetic				320
4	(c) Definability of Turing Machine Computations				
	in Fragments of Bounded Arithmetic				330
	(d) Witnessing Functions				337
	(e) On the Finite Axiomatizability of Bounded Arithmetic				350
5.	Interpretability and Consistency				360
	(a) Introduction				360
	(b) Truth Definitions for Bounded Formulae				361
	(c) An Interpretation of $I\Sigma_0$ in Q				366
	(d) Cut-Elimination and Herbrand's Theorem				
	in Bounded Arithmetic				371
	(e) The Π_1 Theorems of $I\Sigma_0 + Exp$				380
	(f) Incompleteness Theorems				386
	(g) On the Limited Use of Exponentiation				393
B	ibliographical Remarks and Further Reading		•	€.	397
В	ibliography				409
				-	
In	dex of Terms	•	•	•	455
Ir	idex of Symbols		•		459