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Algorithmic Topology and Classification of 3-Manifolds

Second Edition

With 264 Figures and 36 Tables

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