

Contents

Part I. Basic Definitions and Properties

1. The Basic Equations	3
1.1 The Acoustics Equation	3
1.2 The Maxwell Equations	4
1.2.1 The 3D Case	4
1.2.2 The 2D Case	6
1.3 The Elastics System	7
1.3.1 General Formulation	7
1.3.2 The Isotropic Case	8
1.4 Boundary Conditions	11
1.4.1 The Wave Equation	11
1.4.2 The Maxwell Equations	12
1.4.3 The Elastics System	12
2. Functional Issues	15
2.1 Some Functional Spaces	15
2.1.1 Sobolev Spaces	15
2.1.2 Spaces $H(\mathbf{curl}, \Omega)$ and $H(\mathbf{div}, \Omega)$	17
2.2 Variational Formulations	18
2.2.1 The Acoustics Equation	18
2.2.2 The Maxwell Equations	20
2.2.3 The Elastics System	22
2.3 Energy Identities	22
2.3.1 The Acoustics Equation	23
2.3.2 The Maxwell Equations	23
2.3.3 The Elastics System	24
3. Plane Wave Solutions	25
3.1 A General Solution of the Homogeneous Wave Equation	25
3.2 Application to the Maxwell Equations	26
3.2.1 The 3D Case	26
3.2.2 The 2D Case	28
3.3 Application to the Elastics System	29

Part II. Finite Difference Methods

4. Construction of the Schemes in Homogeneous Media	35
4.1 A Model Problem	35
4.2 Second-Order Approximation in Space	35
4.2.1 The 1D Case	35
4.2.2 The 2D Case	38
4.3 Fourth-Order Approximations in Space	41
4.3.1 First Approach: Global Approximation of Δ	41
4.3.2 Second Approach: Fourth-Order Approximations of the First-Order Operators	43
4.4 Approximation in Time	45
4.4.1 The Modified Equation Approach	46
4.4.2 Symmetric Schemes	48
4.5 Higher-Order Approximations in Space	50
4.5.1 First Approach	50
4.5.2 Second Approach	52
4.5.3 Extension to Higher Dimensions	54
4.6 Higher-Order Approximations in Time	56
4.6.1 The Modified Equation Approach	56
4.6.2 Symmetric Schemes	57
4.7 Extension to Systems	58
4.7.1 The Maxwell Equations	58
4.7.2 The Elastics System	60
4.7.3 Higher-Order Approximation in Time	62
5. The Dispersion Relation	65
5.1 Second-Order Schemes for the Wave Equation	65
5.1.1 Using Plane Wave Solutions	65
5.1.2 Computation by the Discrete Fourier Transform	66
5.1.3 Symbol of an Operator	68
5.2 Higher-Order Approximations in Space	69
5.2.1 The First Approach	69
5.2.2 The Second Approach	72
5.3 Approximation in Time	74
5.3.1 Second-Order Approximation in Time	74
5.3.2 The Modified Equation Approach	75
5.3.3 Symmetric Schemes	76
5.4 The Case of Systems	78
5.4.1 The Maxwell Equations	78
5.4.2 The Elastics System	80

6. Stability of the Schemes	83
6.1 General Presentation	83
6.2 Positivity of an Operator	84
6.3 Second-Order Approximation in Time	85
6.3.1 Second-Order Approximation in Space	86
6.3.2 A Basic Property	86
6.3.3 Application to Higher-Order Approximation in Space	87
6.4 The Modified Equation Approach	89
6.4.1 Preliminary Results	89
6.4.2 Fourth-Order Approximation in Space: First Approach	90
6.4.3 Fourth-Order Approximation in Space: Second Approach	92
6.5 Symmetric Schemes	95
6.5.1 First Method	95
6.5.2 Second Method	97
6.6 The Case of Systems	97
6.7 A Numerical Illustration	99
7. Numerical Dispersion and Anisotropy	101
7.1 Phase and Group Velocities	101
7.2 The Concept of Numerical Dispersion	102
7.3 Order of the Numerical Dispersion	103
7.3.1 Schemes Semi-Discrete in Space	104
7.3.2 Fully Discrete Schemes: Second-Order in Time	104
7.3.3 Fully Discrete Schemes: The Modified Equation Approach	106
7.3.4 Fully Discrete Schemes: The Fourth-Order Symmetric Scheme	106
7.3.5 Error Committed on the Group Velocities	108
7.4 Change of Variables	109
7.5 Some Useful Properties of the Schemes	111
7.5.1 Relation between 1D and Higher Dimensions	111
7.5.2 Two Remarkable Schemes	113
7.6 Dispersion Curves	114
7.6.1 Second and Fourth-Order Schemes, Semi-Discrete in Space	114
7.6.2 Schemes, Second-Order in Time and Fourth-Order in Space	114
7.6.3 Schemes, Fourth-Order in Time and Space	115
7.6.4 Comparison with Higher-Order Schemes in Space	118
7.7 Isotropy Curves	118
7.8 The Elastics System	118

8. Construction of the Schemes in Heterogeneous Media	123
8.1 A General Framework	123
8.2 Second-Order Approximations	123
8.3 Higher-Order Approximations: First Approach	124
8.4 Higher-Order Approximations: Second Approach	126
8.5 The Case of Arakawa's Scheme	128
8.6 Approximation in Time	129
8.6.1 Second-Order Approximation in Time	129
8.6.2 The Heterogeneous Modified Equation	130
8.6.3 Expression of the Correction Term	131
8.7 Approximation of the Boundary Conditions	133
8.7.1 Second-Order Schemes	133
8.7.2 Higher-Order Schemes	134
8.7.3 Extension to Systems	136
9. Stability by Energy Techniques	137
9.1 Positivity of the Discrete Operators	137
9.1.1 Second-Order and Second Approach for Fourth-Order Approximations	137
9.1.2 Fourth-Order: First Approach	138
9.2 Stability Conditions	142
9.2.1 A General Framework	142
9.2.2 Computation of $\ A_h\ $ for the Second-Order Approximation	143
10. Reflection-Transmission Analysis	145
10.1 The 1D Case	145
10.1.1 The Continuous Problem	145
10.1.2 Second-Order Approximation	146
10.1.3 Fourth-Order: First Approach	147
10.1.4 A Numerical Study	153
10.2 The 2D Case	156
10.2.1 The Continuous Problem	156
10.2.2 Second-Order Scheme	158
10.2.3 A Numerical Experiment	162

Part III. Finite Element Methods

11. Mass-Lumping in 1D	169
11.1 Basic Approximations	169
11.1.1 Construction of Mass-Lumped Finite Elements	169
11.1.2 Approximation in Time	177
11.2 Dispersion Relations	179
11.2.1 P_2 Finite Elements	179
11.2.2 P_3 and Higher-Order Finite Elements	181

11.3	Stability Analysis	186
11.3.1	The Leapfrog Scheme	186
11.3.2	The Modified Equation Approach	189
11.3.3	Symmetric Schemes	191
11.4	Dispersion Analysis	192
11.4.1	Taylor Expansions	192
11.4.2	Dispersion Curves	193
11.5	Some Results on the Amplitudes	194
11.5.1	Error Estimates on the Physical Part of the Solution	200
11.5.2	Error Estimates on the Parasitic Part of the Solution	202
11.6	Reflection-Transmission Analysis	202
11.6.1	FEM Approximation of the Heterogeneous Wave Equation	203
11.6.2	Taylor Expansion of the Wavenumber	203
11.6.3	Interface Between Two Elements	204
11.6.4	Interface at an Interior Point	205
11.6.5	Extension to Higher-Order Approximations	206
11.7	Taylor Expansions of the Eigenvectors	208
12.	Spectral Elements	211
12.1	Construction of Quadrilateral and Hexahedral Finite Elements	211
12.1.1	Reference Spectral Elements	211
12.1.2	Extension to Quadrilateral Meshes	214
12.1.3	Extension to Hexahedral Meshes	220
12.2	Plane Wave Analysis of Regular Meshes	222
12.2.1	Decomposition of the Discrete Equations	222
12.2.2	Decomposition of the Eigenvalues and Eigenvectors	225
12.3	Some Analysis of Non-Regular Meshes in 2D	230
12.3.1	Dispersion Analysis	230
12.3.2	Numerical Study of the Stability	231
12.3.3	Numerical Study of the Accuracy	234
12.3.4	A Two-Layer Experiment	238
12.4	Triangular and Tetrahedral Meshes	244
12.4.1	The Basic Problem	244
12.4.2	A New Family of Triangular Elements	250
12.4.3	Tetrahedral Elements	254
12.4.4	Non-Conforming Triangular Elements	257
12.5	A Numerical Illustration	259
13.	Mass-Lumped Mixed Formulations and Edge Elements	261
13.1	Variational Extensions of the Yee Scheme	261
13.1.1	The Model Problem	261
13.1.2	A First Family of Hexahedral Edge Elements	262
13.1.3	The 2D Case	269

13.2	Efficient Edge Elements for the Maxwell Equations	271
13.2.1	Extension to Anisotropic Media and Complex Geometries	271
13.2.2	New Spaces of Approximation	273
13.2.3	Basis Functions and Degrees of Freedom	274
13.2.4	The Mass Integral	277
13.2.5	The Stiffness Integral	279
13.2.6	An Efficient Alternative	280
13.2.7	The 2D Case	283
13.2.8	A 2D Numerical Experiment	284
13.3	Triangular and Tetrahedral Edge Elements	284
13.3.1	Triangular Edge Elements	286
13.3.2	Tetrahedral Edge Elements	289
13.3.3	Spaces of Approximation	292
13.4	A New Formulation of Spectral Elements	294
13.4.1	A New Approximation of the Wave Equation	294
13.4.2	A Theorem of Equivalence	296
13.4.3	Extension to the Elasticity System	301
14.	Modeling Unbounded Domains	307
14.1	History of the Problem	307
14.2	Perfectly Matched Layers	308
14.2.1	Presentation of the Method	308
14.2.2	Construction of the PML in 2D	311
14.2.3	The Three-Dimensional Case	317
14.2.4	Finite Element Approximation	320
14.2.5	Approximation in Time	321
14.3	Numerical Illustrations	322
14.3.1	PML for the 2D Elasticity System	322
14.3.2	Scattering by an Ogive of an Electromagnetic Wave	323
14.3.3	A “Foothills” Experiment	324
14.4	Computational Issues	329
14.4.1	Tetrahedra or Hexahedra?	329
14.4.2	Finite Element or Finite Difference Methods?	330
A.	Appendix: Construction of a General $H(\text{curl})$-Conforming Transform	331
A.1	A Local $H(\mathbf{curl}, \hat{K})$ -Conforming Isomorphism	331
A.1.1	Notation	331
A.1.2	A Local $H(\mathbf{curl}, \hat{K})$ -Conforming Isomorphism	332
A.2	A Global $H(\mathbf{curl}, \Omega)$ -Conforming Isomorphism	336
A.2.1	Notation	336
A.2.2	A Global $H(\mathbf{curl}, \Omega)$ -Conforming Isomorphism	337
	Bibliography	341
	Index	347