

# Contents

## Preface — V

## Part I: Basic concepts and principles

### 1 Outline of the book — 5

- 1.1 Part I: Basic concepts and principles — 6
- 1.2 Part II: The meaning of quantum mechanics — 7
- 1.3 Part III: The symmetries of space and time — 11
- 1.4 Part IV: Conservative classical mechanics — 13
- 1.5 Part V: The dynamics of quantum systems — 14
- 1.6 Part VI: Representations of groups and Lie algebras — 15

### 2 Continuous and infinitesimal symmetries — 18

- 2.1 Symmetries — 18
  - 2.1.1 A bit of history — 18
  - 2.1.2 What is a symmetry? — 19
- 2.2 Matrices — 20
- 2.3 Basics on Lie algebras — 23
  - 2.3.1 Lie algebras — 23
  - 2.3.2 Isomorphisms and commutation relations — 25
  - 2.3.3 Matrix Lie algebras — 26
- 2.4 Further concepts for Lie algebras — 28
  - 2.4.1 Some constructions — 28
  - 2.4.2 Universal enveloping algebra and Casimirs — 29
- 2.5 Groups — 31
  - 2.5.1 Group homomorphisms and semidirect products — 33
  - 2.5.2 Group representations — 35
  - 2.5.3 Further concepts for groups — 36
- 2.6 Matrix groups and continuous motions — 36
  - 2.6.1 Matrix groups — 36
  - 2.6.2 Motions — 37
  - 2.6.3 Infinitesimal motions — 38
  - 2.6.4 The identity subgroup — 39
  - 2.6.5 Uniform motions and the matrix exponential — 40
  - 2.6.6 The universal covering group — 42
  - 2.6.7 Discrete groups — 43

### 3 Classical mechanics and quantum mechanics — 44

- 3.1 A hierarchical view of physics — 44

|       |                                             |           |
|-------|---------------------------------------------|-----------|
| 3.1.1 | Units —                                     | <b>44</b> |
| 3.1.2 | Length scales —                             | <b>46</b> |
| 3.1.3 | The basic limits of physics —               | <b>47</b> |
| 3.1.4 | The main hierarchy of fields of physics —   | <b>48</b> |
| 3.2   | Classical mechanics —                       | <b>49</b> |
| 3.2.1 | Particle mechanics —                        | <b>49</b> |
| 3.2.2 | Continuum mechanics —                       | <b>50</b> |
| 3.2.3 | Relativity theory —                         | <b>51</b> |
| 3.2.4 | Statistical mechanics and thermodynamics —  | <b>53</b> |
| 3.2.5 | Hamiltonian mechanics —                     | <b>56</b> |
| 3.3   | Quantum mechanics —                         | <b>58</b> |
| 3.3.1 | The correspondence principle —              | <b>58</b> |
| 3.3.2 | The Schrödinger picture —                   | <b>59</b> |
| 3.3.3 | Bras and kets —                             | <b>62</b> |
| 3.3.4 | The Heisenberg picture —                    | <b>65</b> |
| 3.3.5 | Interaction pictures —                      | <b>68</b> |
| 3.3.6 | The canonical ensemble —                    | <b>68</b> |
| 3.3.7 | Quantum field theory —                      | <b>69</b> |
| 3.3.8 | Deterministic and statistical predictions — | <b>70</b> |

## Part II: The meaning of quantum mechanics

|          |                                                           |            |
|----------|-----------------------------------------------------------|------------|
| <b>4</b> | <b>Measurement in quantum mechanics —</b>                 | <b>77</b>  |
| 4.1      | The quantum measurement process —                         | <b>77</b>  |
| 4.1.1    | Quantum sources and states —                              | <b>78</b>  |
| 4.1.2    | Measurement devices and quantum measures —                | <b>82</b>  |
| 4.2      | The statistical interpretation —                          | <b>85</b>  |
| 4.2.1    | Statistical expectations —                                | <b>85</b>  |
| 4.2.2    | The statistical interpretation of quantum mechanics —     | <b>87</b>  |
| 4.2.3    | The spectral version of Born's rule —                     | <b>88</b>  |
| 4.3      | Quantum tomography —                                      | <b>91</b>  |
| 4.3.1    | Quantum measurement tomography and POVMs —                | <b>92</b>  |
| 4.3.2    | Informationally complete quantum measures —               | <b>93</b>  |
| 4.3.3    | Quantum state tomography —                                | <b>93</b>  |
| 4.3.4    | Completely positive maps and quantum process tomography — | <b>95</b>  |
| 4.3.5    | Nonstationary sources —                                   | <b>100</b> |
| 4.3.6    | Quantum tomography of instruments —                       | <b>102</b> |
| 4.3.7    | Composite sources, filters, and detectors —               | <b>103</b> |
| <b>5</b> | <b>Quantum dynamics and spectroscopy —</b>                | <b>105</b> |
| 5.1      | Quantum dynamics —                                        | <b>105</b> |

|          |                                                        |            |
|----------|--------------------------------------------------------|------------|
| 5.1.1    | Quantum beams and quantum Liouville equation —         | 105        |
| 5.1.2    | Quantum phase space —                                  | 106        |
| 5.1.3    | The Ehrenfest picture —                                | 108        |
| 5.1.4    | Pure state idealization and the Schrödinger equation — | 109        |
| 5.1.5    | Quantum dynamics in mixing linear media —              | 111        |
| 5.1.6    | Nonlinear dissipative quantum dynamics —               | 113        |
| 5.2      | Spectroscopy and system identification —               | 114        |
| 5.2.1    | Energy spectrum and quantum fluctuations —             | 114        |
| 5.2.2    | Spectroscopy and high precision measurements —         | 116        |
| 5.2.3    | Example: The gyromagnetic ratio of the electron —      | 118        |
| 5.3      | Quantum fields and macroscopic quantum systems —       | 121        |
| 5.3.1    | Extended sources —                                     | 121        |
| 5.3.2    | The emergence of quantum fields —                      | 122        |
| 5.3.3    | Macroscopic quantum systems —                          | 123        |
| <b>6</b> | <b>On the foundations of quantum mechanics —</b>       | <b>125</b> |
| 6.1      | Measurement uncertainty —                              | 125        |
| 6.1.1    | Statistical uncertainty —                              | 125        |
| 6.1.2    | Imperfect measurements —                               | 128        |
| 6.1.3    | Measurement errors —                                   | 131        |
| 6.2      | Quantum measures for particular detectors —            | 131        |
| 6.2.1    | Beam splitters and polarization state measurements —   | 132        |
| 6.2.2    | Joint measurements of noncommuting quantities —        | 134        |
| 6.2.3    | Realistic measurements of position —                   | 135        |
| 6.2.4    | Measuring particle tracks —                            | 135        |
| 6.3      | On the interpretation of quantum mechanics —           | 136        |
| 6.3.1    | The origin of discrete randomness —                    | 136        |
| 6.3.2    | The quantum measurement problem —                      | 139        |
| 6.3.3    | Particles and fields —                                 | 142        |
| 6.3.4    | The universe as a quantum system —                     | 143        |
| 6.3.5    | Reproducibility and objectivity —                      | 144        |
| 6.3.6    | Knowledge and probability —                            | 145        |
| 6.3.7    | The thermal interpretation of quantum physics —        | 146        |

## Part III: The symmetries of space and time

|          |                                       |            |
|----------|---------------------------------------|------------|
| <b>7</b> | <b>Rotations and translations —</b>   | <b>155</b> |
| 7.1      | Rotations and $SO(n)$ —               | 155        |
| 7.1.1    | 3-vectors —                           | 158        |
| 7.1.2    | 3-dimensional rotations and $SO(3)$ — | 160        |
| 7.1.3    | The exponential map —                 | 164        |

|           |                                                             |            |
|-----------|-------------------------------------------------------------|------------|
| 7.2       | Pauli matrices and quaternions —                            | <b>165</b> |
| 7.2.1     | Pauli matrices —                                            | <b>165</b> |
| 7.2.2     | Pauli 4-vectors —                                           | <b>166</b> |
| 7.2.3     | Quaternions and rotations —                                 | <b>168</b> |
| 7.3       | Rigid motion and angular momentum —                         | <b>173</b> |
| 7.3.1     | Rigid motions and Euclidean groups —                        | <b>173</b> |
| 7.3.2     | Rotating rigid bodies —                                     | <b>174</b> |
| 7.3.3     | Rotations and angular momentum —                            | <b>177</b> |
| 7.3.4     | Classical rigid body dynamics —                             | <b>180</b> |
| <b>8</b>  | <b>Unitary groups: Few-level systems, qubits, quarks —</b>  | <b>182</b> |
| 8.1       | $N$ -level systems and $U(N)$ —                             | <b>182</b> |
| 8.1.1     | Atomic energy levels —                                      | <b>182</b> |
| 8.1.2     | $N$ -level systems —                                        | <b>183</b> |
| 8.2       | Rotations and $SU(2)$ —                                     | <b>186</b> |
| 8.2.1     | The isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ — | <b>186</b> |
| 8.2.2     | Qubits and Bloch sphere —                                   | <b>187</b> |
| 8.2.3     | Polarized light and beam transformations —                  | <b>189</b> |
| 8.2.4     | Historical remarks —                                        | <b>191</b> |
| 8.2.5     | Light and Maxwell equations —                               | <b>192</b> |
| 8.2.6     | The Stern–Gerlach experiment —                              | <b>194</b> |
| 8.3       | The quark model and $SU(3)$ —                               | <b>196</b> |
| <b>9</b>  | <b>Lie groups and Lie algebras —</b>                        | <b>199</b> |
| 9.1       | Motions on vector spaces and Lie algebras —                 | <b>199</b> |
| 9.1.1     | Motions on vector spaces —                                  | <b>199</b> |
| 9.1.2     | Lie products and Lie coproducts —                           | <b>201</b> |
| 9.1.3     | Some Lie algebra constructions —                            | <b>204</b> |
| 9.1.4     | Adjoint and coadjoint representation —                      | <b>206</b> |
| 9.2       | Linear groups and their Lie algebras —                      | <b>208</b> |
| 9.2.1     | Time-ordered exponential and uniform motions —              | <b>209</b> |
| 9.2.2     | The Lie algebra of infinitesimal motions —                  | <b>211</b> |
| 9.2.3     | Motions in infinite dimensions —                            | <b>213</b> |
| 9.3       | Lie algebras from derivations —                             | <b>214</b> |
| 9.4       | Lie $\ast$ -algebras —                                      | <b>217</b> |
| 9.4.1     | Complexification —                                          | <b>217</b> |
| 9.4.2     | Linear Lie groups —                                         | <b>219</b> |
| <b>10</b> | <b>Important families of Lie algebras —</b>                 | <b>221</b> |
| 10.1      | Classical Lie groups and their Lie algebras —               | <b>221</b> |
| 10.1.1    | The general linear group $GL(n, \mathbb{K})$ —              | <b>221</b> |
| 10.1.2    | The special linear group $SL(n, \mathbb{K})$ —              | <b>222</b> |

|           |                                                          |            |
|-----------|----------------------------------------------------------|------------|
| 10.1.3    | Groups and Lie algebras preserving bilinear forms —      | 223        |
| 10.1.4    | The orthogonal groups $O(n, \mathbb{K})$ and $O(p, q)$ — | 224        |
| 10.1.5    | The groups $SO(p, q)$ and $ISO(p, q)$ —                  | 225        |
| 10.1.6    | The symplectic group $Sp(2n, \mathbb{K})$ —              | 226        |
| 10.1.7    | The unitary groups $U(n)$ and $SU(n)$ —                  | 226        |
| 10.2      | Finite-dimensional semisimple Lie algebras —             | 228        |
| 10.3      | Composite Lie algebras and Lie groups —                  | 231        |
| 10.3.1    | Heisenberg algebras —                                    | 231        |
| 10.3.2    | Heisenberg groups —                                      | 234        |
| 10.3.3    | The Heisenberg extension of a Lie algebra —              | 236        |
| <b>11</b> | <b>Symmetries of spacetime —</b>                         | <b>238</b> |
| 11.1      | Minkowski spacetime —                                    | 238        |
| 11.1.1    | Minkowski space —                                        | 239        |
| 11.1.2    | Causality —                                              | 240        |
| 11.1.3    | General spacetime —                                      | 242        |
| 11.1.4    | The Poincaré group —                                     | 243        |
| 11.2      | Local Lorentz transformations —                          | 244        |
| 11.2.1    | The family of Lorentz groups —                           | 244        |
| 11.2.2    | Lorentz transformations of special form —                | 248        |
| 11.2.3    | Orbits of the proper Lorentz group —                     | 249        |
| 11.2.4    | Infinitesimal Lorentz transformations —                  | 250        |
| 11.3      | The 4-dimensional case —                                 | 253        |
| 11.3.1    | $SL(2, \mathbb{C})$ as doubled Lorentz group —           | 253        |
| 11.3.2    | Covering the discrete symmetries $P$ and $T$ —           | 255        |
| 11.3.3    | The Lorentz Lie algebra $so(1, 3)$ —                     | 255        |
| 11.3.4    | The Poincaré group $ISO(1, 3)$ —                         | 258        |
| 11.3.5    | The doubled Poincaré group —                             | 258        |
| 11.4      | The nonrelativistic limit —                              | 259        |
| 11.4.1    | Galilean spacetime and the Galilean group —              | 261        |

## Part IV: Conservative classical mechanics

|           |                                                   |            |
|-----------|---------------------------------------------------|------------|
| <b>12</b> | <b>Classical oscillating systems —</b>            | <b>267</b> |
| 12.1      | Systems of damped oscillators —                   | 267        |
| 12.1.1    | Multimodal potentials —                           | 270        |
| 12.1.2    | Phase space formulation —                         | 271        |
| 12.1.3    | The classical anharmonic oscillator —             | 271        |
| 12.1.4    | The anharmonic oscillator in phase space —        | 273        |
| 12.2      | The classical harmonic oscillator —               | 275        |
| 12.2.1    | Harmonic oscillators and linear field equations — | 276        |

- 12.2.2 Alpha rays — **278**
- 12.2.3 Beta rays — **281**
- 12.2.4 Light rays and gamma rays — **283**

### **13 Conservative classical mechanics — 287**

- 13.1 Poisson algebras for classical mechanics — **287**
  - 13.1.1 Poisson algebras — **287**
  - 13.1.2 Lie–Poisson algebras — **290**
  - 13.1.3 Poisson representations of symplectic Lie algebras — **294**
  - 13.1.4 Classical symplectic mechanics — **296**
- 13.2 Commutative Poisson algebras from closed 2-forms — **299**
  - 13.2.1 The symplectic case — **301**
- 13.3 Classical conservative Hamiltonian dynamics — **303**
  - 13.3.1 Unconstrained Hamiltonian dynamics — **303**
  - 13.3.2 Constrained Hamiltonian dynamics — **306**
  - 13.3.3 A Hamiltonian system with gauge symmetry — **308**
  - 13.3.4 A variational principle — **309**
- 13.4 Classical Lagrangian mechanics — **310**
- 13.5 Molecular mechanics — **311**

## **Part V: The dynamics of quantum systems**

### **14 Hamiltonian quantum mechanics — 319**

- 14.1 Hermitian spaces — **319**
  - 14.1.1 Hermitian spaces and their antidual — **320**
  - 14.1.2 Norm and completion of a Hermitian space — **322**
  - 14.1.3 Linear mappings between Hermitian spaces — **325**
- 14.2 Quantum dynamics as symplectic motion — **327**
  - 14.2.1 The Dirac–Frenkel variational principle — **328**
- 14.3 Quantum-classical dynamics — **330**
  - 14.3.1 Examples — **332**
  - 14.3.2 Quantum-classical dynamics in the Heisenberg picture — **333**
- 14.4 Quantization — **334**
  - 14.4.1 Berezin quantization and geometric quantization — **334**
  - 14.4.2 Deformation quantization — **334**
  - 14.4.3 The Wigner transform — **337**
  - 14.4.4 The symbol formulation — **342**

### **15 Quantum oscillators and coherent states — 346**

- 15.1 Quantizing the harmonic oscillator — **346**
  - 15.1.1 Extension to the anharmonic case — **348**

|                                                                |                                                               |            |
|----------------------------------------------------------------|---------------------------------------------------------------|------------|
| 15.1.2                                                         | The oscillator algebra $os(1)$ —                              | 349        |
| 15.1.3                                                         | Representations of the Heisenberg algebra —                   | 350        |
| 15.2                                                           | The single-mode boson Fock space —                            | 352        |
| 15.3                                                           | Coherent states for the harmonic oscillator —                 | 355        |
| 15.3.1                                                         | Group coherent states —                                       | 355        |
| 15.3.2                                                         | Glauber coherent states —                                     | 356        |
| 15.3.3                                                         | Resolution of unity —                                         | 357        |
| 15.3.4                                                         | The Bargmann representation —                                 | 359        |
| 15.3.5                                                         | Monochromatic beams and coherent states —                     | 362        |
| <b>16</b>                                                      | <b>Dynamical equations for quantum systems —</b>              | <b>363</b> |
| 16.1                                                           | Hamiltonians —                                                | 363        |
| 16.1.1                                                         | Hamiltonians in an $N$ -level system —                        | 366        |
| 16.2                                                           | The quantum spectrum —                                        | 367        |
| 16.2.1                                                         | The spectral theorem: informal version —                      | 368        |
| 16.2.2                                                         | The spectral theorem: uses —                                  | 370        |
| 16.2.3                                                         | The spectral theorem: semiformal version —                    | 371        |
| 16.2.4                                                         | The spectral theorem: rigorous version —                      | 372        |
| 16.3                                                           | Solving the Heisenberg, Liouville, and Schrödinger equation — | 374        |
| 16.3.1                                                         | Dynamics in the Heisenberg picture —                          | 375        |
| 16.3.2                                                         | The quantum Liouville equation —                              | 376        |
| 16.3.3                                                         | Dynamics in the Schrödinger picture —                         | 376        |
| <br><b>Part VI: Representations of groups and Lie algebras</b> |                                                               |            |
| <b>17</b>                                                      | <b>Representation theory —</b>                                | <b>383</b> |
| 17.1                                                           | Representations of matrix Lie groups —                        | 383        |
| 17.1.1                                                         | Linear representations —                                      | 383        |
| 17.1.2                                                         | Projective representations —                                  | 384        |
| 17.2                                                           | Representations of Lie algebras —                             | 386        |
| 17.2.1                                                         | General representations —                                     | 386        |
| 17.2.2                                                         | Triangular decompositions —                                   | 388        |
| 17.2.3                                                         | The semisimple case —                                         | 389        |
| 17.2.4                                                         | Triangulated Lie algebras of rank and degree one —            | 391        |
| 17.2.5                                                         | Classification for rank and degree one —                      | 392        |
| 17.2.6                                                         | Unitary highest weight representations —                      | 392        |
| <b>18</b>                                                      | <b>Representations for quantum particles —</b>                | <b>395</b> |
| 18.1                                                           | Linear representations of $GL(2, \mathbb{C})$ and $SU(2)$ —   | 395        |
| 18.1.1                                                         | Spin —                                                        | 395        |
| 18.1.2                                                         | Pauli sets —                                                  | 396        |

|                        |                                                      |            |
|------------------------|------------------------------------------------------|------------|
| 18.1.3                 | Tridiagonal form —                                   | <b>398</b> |
| 18.1.4                 | The spinor representations of $GL(2, \mathbb{C})$ —  | <b>399</b> |
| 18.2                   | Spin coherent states —                               | <b>401</b> |
| 18.2.1                 | The completeness relation —                          | <b>402</b> |
| 18.3                   | Elementary particles —                               | <b>404</b> |
| 18.3.1                 | Mass and spin —                                      | <b>406</b> |
| 18.3.2                 | Casimirs —                                           | <b>407</b> |
| 18.4                   | Physical representations of the Poincaré group —     | <b>408</b> |
| 18.4.1                 | The spinor representations —                         | <b>409</b> |
| 18.4.2                 | Definiteness —                                       | <b>411</b> |
| 18.4.3                 | Lorentz invariant measures —                         | <b>412</b> |
| <b>19</b>              | <b>Spectra and spectroscopy —</b>                    | <b>413</b> |
| 19.1                   | Probing the spectrum of a system —                   | <b>414</b> |
| 19.2                   | From the early history of quantum mechanics —        | <b>417</b> |
| 19.2.1                 | Spectral lines —                                     | <b>417</b> |
| 19.2.2                 | The hydrogen spectrum —                              | <b>418</b> |
| 19.2.3                 | Nobel prizes for the pioneers of quantum mechanics — | <b>421</b> |
| 19.2.4                 | Black body radiation —                               | <b>422</b> |
| 19.2.5                 | Einstein's derivation of Planck's law —              | <b>425</b> |
| 19.2.6                 | Modern derivation of Planck's law —                  | <b>428</b> |
| 19.2.7                 | Stefan's law and Wien's displacement law —           | <b>430</b> |
| 19.3                   | Spectra of systems of particles —                    | <b>432</b> |
| 19.3.1                 | Examples of spectra —                                | <b>434</b> |
| 19.3.2                 | Spectrum deformation and avoided crossings —         | <b>436</b> |
| 19.4                   | Dynamical symmetries —                               | <b>439</b> |
| 19.4.1                 | Symmetries and broken symmetries —                   | <b>439</b> |
| 19.5                   | Chains of subalgebras —                              | <b>441</b> |
| 19.5.1                 | Branching rules —                                    | <b>443</b> |
| 19.5.2                 | Clebsch–Gordan coefficients —                        | <b>444</b> |
| 19.6                   | The hydrogen atom —                                  | <b>445</b> |
| 19.6.1                 | Symmetries of the hydrogen atom —                    | <b>446</b> |
| 19.6.2                 | Symmetry breaking in the hydrogen atom —             | <b>447</b> |
| 19.6.3                 | Kepler's laws, the hydrogen atom, and $SO(4)$ —      | <b>448</b> |
| <b>Bibliography —</b>  |                                                      | <b>453</b> |
| <b>Author index —</b>  |                                                      | <b>465</b> |
| <b>Symbol index —</b>  |                                                      | <b>469</b> |
| <b>Subject index —</b> |                                                      | <b>471</b> |