

Contents

1	Introduction	1
1.1	Statement of the main results	2
1.2	Background	3
1.3	Outline of the paper	6
1.4	Conventions and notation	8
2	Unstable polynomials of algebraic surfaces	12
2.1	Introduction	12
2.1.1	Generalities on the ν -map	14
2.2	A stratification of parameter spaces for vector bundles on $\tilde{S}$	15
2.2.1	An inductive procedure that defines the type of a bundle near E	16
2.2.2	Definition of the stratification by type near E	18
2.2.3	The pushforward to S	19
2.3	The stratification of $\mathcal{M}_{c+k}(\tilde{S}, \tilde{H})$	19
2.3.1	The case of polarizations near to π^*H	20
2.3.2	The morphism from $X^{t,st}$ to $\mathcal{M}_{c+k- t }(S, H)$	21
2.4	The Λ_t construction	21
2.4.1	The construction of $\Lambda_t(\mathcal{V})$	22
2.4.2	Proof of Proposition 2.4.1	24
2.5	Analysis of the strata of $\mathcal{M}_{c+k}(\tilde{S}, \tilde{H}(\mathbf{r}))$	25
2.5.1	The strata $X^{t,st}$	25
2.5.2	The strata $X^{t,ss}$	26
2.6	Proofs of the theorems.	27
2.6.1	Proof of Theorem 2.1.1	27
2.6.2	Relative moduli spaces	28
2.6.3	The relative ν map and the invariance of δ	30
2.6.4	Proof of Theorem 2.1.2	32
3	Identification of $\delta_{3,r}(S, H)$ with $\gamma_3(S)$	33
3.1	The main results	33
3.2	The family of $K3$ surfaces with a section	35
3.2.1	The period space and the global Torelli theorem.	35
3.2.2	Construction of the family	37
3.3	The family of minimal elliptic surfaces with multiple fibers	40
3.3.1	Construction of the family	40
3.3.2	Relationship between the cohomology of an elliptic fibration and that of its jacobian surface	42
3.3.3	Analogue of Theorem 3.2.10 for the family of elliptic surfaces with multiple fibers.	46
3.4	The family of blown up elliptic surfaces	49
3.5	Proof of Theorem 3.1.4	50
3.5.1	The generic subset of $\tilde{T}$	51
3.5.2	The Invariant theory argument	55

4	Certain moduli spaces for bundles on elliptic surfaces with $p_g = 1$	57
4.1	Background material on extensions of rank one sheaves	58
4.2	The parameter spaces for properly semi-stable bundles	59
4.3	The moduli spaces $\mathcal{M}_c(S, H)$ for $1 \leq c \leq 3$.	64
4.3.1	A description of V as an extension	64
4.3.2	The parameter spaces for vertical extensions	67
4.3.3	Computation of dimensions of cohomology groups	69
4.3.4	The dimension of $\mathcal{M}_c(S, H)$	70
4.4	Irreducible components of $\overline{\mathcal{M}}_3(S, H)$ associated to large divisors	73
4.5	Four-dimensional components of $\mathcal{M}_2(S, H)$	79
4.5.1	A more detailed study of $\mathcal{M}_2^D(S, H)$	82
4.6	Multiplicities.	86
4.6.1	The versal deformation space of a vector bundle	87
4.6.2	Proof of Theorem 4.6.1	88
4.6.3	Proof of Theorem 4.6.2	93
4.6.4	Proof of Theorem 4.6.3	95
4.7	Definition of $\delta_3^{\text{st}}(S, H)$ and $\delta_3^{\text{ss}}(S, H)$	95
4.7.1	$\delta_3^{\text{st}}(S, H)$	96
4.7.2	The line bundle M_0 over $\overline{\mathcal{P}}_{3,L}(S, H)$ in the case $L^2 \cong \mathcal{O}_S$.	96
4.7.3	$\delta_3^{\text{ss}}(S, H)$	97
5	Representatives for classes in the image of the ν-map	99
5.1	Representatives for the ν map	99
5.1.1	Generalities on Chern classes	99
5.1.2	A divisor representing $\nu([C])$	99
5.1.3	Holomorphic 2-form representatives	101
5.1.4	The divisors Δ and two-forms λ on $\mathcal{M}_c(S, H)$ and $P_c(S, H)$	103
5.1.5	Elementary properties of $\Delta_{\mathcal{F}}(C, L)$ and $\lambda_{\mathcal{F}}(\omega)$	105
5.2	Passage from the blow-up to the original surface	106
5.2.1	Relation between the ν -map for S and $\tilde{S}$	106
5.2.2	Avoiding base-points	107
5.3	Enumerative Geometry	107
5.4	$\epsilon_2(S, H)$	111
6	The blow-up formula	112
6.1	Outline of the proof of Theorem 6.0.1 for $k = 2$	113
6.2	First results	116
6.2.1	The basic properties of the divisors $\Delta(E_1)$ and $\Delta(E_2)$	116
6.2.2	The definition of $Y(X)$, $c(X)$, $d(X)$.	117
6.2.3	Comparing classes after semistable reduction	117
6.3	An extension of the family $\Lambda_t(\mathcal{V})$	120
6.3.1	The basic construction	120
6.3.2	The basic formula	124
6.4	Proof of Proposition 6.1.3	125
6.4.1	Enumerating the components A_j	125
6.4.2	Proof of Proposition 6.4.1 in the case when $Y(A_j) \subset \mathcal{M}_2(S, H)$	128
6.4.3	Proof of Proposition 6.4.1 in the case when $Y(A_j) \subset P_0(S, H)$	129
6.5	The contribution of the X_i	134

6.5.1	Initial cases when the contribution is zero	135
6.5.2	The X_i such that $Y(X_i) \subset \mathcal{M}_3(S, H)$	137
6.5.3	The X_i such that $Y(X_i) \subset \mathcal{M}_2(S, H)$	140
6.5.4	The X_i such that $Y(X_i) \subset P_3(S, H)$	151
6.5.5	The X_i such that $Y(X_i) \subset P_c(S, H)$ with $c \leq 2$	156
6.6	The multiplicity of the X_i such that $\delta(X_i) \neq 0$	160
6.6.1	The scheme Δ^0	160
6.6.2	The case when $Y(X_i) \subset \mathcal{M}_3(S, H)$	162
6.6.3	The case when $Y(X_i) \subset \mathcal{M}_2(S, H)$	163
6.6.4	The case when $Y(X_i) \subset P_3(S, H)$	163
7	The proof of Theorem 1.1.1	167
7.1	Only the components of $\mathcal{M}_3(S, H)$ associated to large divisors contribute to the first two coefficients of $\delta_3^{\text{st}}(S, H)$	169
7.2	The proof of the first part of Proposition 7.0.10	171
7.2.1	The combinatorics of the set R	171
7.3	A further study of the components $\mathcal{M}_3^D(S, H)$	174
7.3.1	Properties of $\mathcal{V}^D$ in the n^{th} -order neighborhood of $F_{i,*}^{(3)}$	176
7.3.2	Certain extensions on S and their properties	178
7.3.3	Some local computations	180
7.3.4	The proof of Proposition 7.3.2	182
7.4	The computation of $c'_1(m_1, m_2)$	189
7.4.1	Reduction to a computation on $\text{Hilb}^3(S)$	190
7.4.2	An expression for $\bar{\nu}_{\mathcal{W}^D}([C])$	192
7.4.3	An expression for $c'_1(m_1, m_2)$ as an integral over R	193
7.4.4	A more explicit expression for $c'_1(m_1, m_2)$	196
7.4.5	Formulas for certain sums over T	199
7.4.6	Completion of the proof of Proposition 7.0.11	201
7.5	Proof of Formula (79) and of Proposition 7.0.12	202
7.5.1	Proof of Proposition 7.5.1	202
7.5.2	Proof of Proposition 7.0.12	208
8	Appendix: The non-simply connected case – by John W. Morgan, Millie Niss and Kieran O’Grady	211
8.1	Proof of Proposition 8.0.20	212
8.2	Proof of Proposition 8.0.21	214
8.3	Computation of $e_2(S, H)$	216
	References	219
	Index	222