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# Geometry IV

Non-regular Riemannian Geometry

With 58 Figures



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# **I. Two-Dimensional Manifolds of Bounded Curvature**

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Translated from the Russian  
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