

Vladimir I. Arnol'd

Ordinary Differential Equations

Translated from the Russian
by Roger Cooke

With 272 Figures

Springer-Verlag

Berlin Heidelberg New York

London Paris Tokyo

Hong Kong Barcelona

Budapest

Contents

Chapter 1. Basic Concepts	13
§ 1. Phase Spaces	13
1. Examples of Evolutionary Processes	13
2. Phase Spaces	14
3. The Integral Curves of a Direction Field	16
4. A Differential Equation and its Solutions	17
5. The Evolutionary Equation with a One-dimensional Phase Space	19
6. Example: The Equation of Normal Reproduction	21
7. Example: The Explosion Equation	23
8. Example: The Logistic Curve	24
9. Example: Harvest Quotas	25
10. Example: Harvesting with a Relative Quota	26
11. Equations with a Multidimensional Phase Space	27
12. Example: The Differential Equation of a Predator-Prey System	28
13. Example: A Free Particle on a Line	31
14. Example: Free Fall	32
15. Example: Small Oscillations	32
16. Example: The Mathematical Pendulum	33
17. Example: The Inverted Pendulum	34
18. Example: Small Oscillations of a Spherical Pendulum	34
§ 2. Vector Fields on the Line	36
1. Existence and Uniqueness of Solutions	36
2. A Counterexample	36
3. Proof of Uniqueness	37
4. Direct Products	39
5. Examples of Direct Products	39
6. Equations with Separable Variables	41
7. An Example: The Lotka-Volterra Model	43
§ 3. Linear Equations	48
1. Homogeneous Linear Equations	48

2. First-order Homogeneous Linear Equations with Periodic Coefficients	49
3. Inhomogeneous Linear Equations	51
4. The Influence Function and δ -shaped Inhomogeneities	53
5. Inhomogeneous Linear Equations with Periodic Coefficients	56
§ 4. Phase Flows	57
1. The Action of a Group on a Set	57
2. One-parameter Transformation Groups	59
3. One-parameter Diffeomorphism Groups	61
4. The Phase Velocity Vector Field	63
§ 5. The Action of Diffeomorphisms on Vector Fields and Direction Fields	66
1. The Action of Smooth Mappings on Vectors	66
2. The Action of Diffeomorphisms on Vector Fields	70
3. Change of Variables in an Equation	72
4. The Action of a Diffeomorphism on a Direction Field	73
5. The Action of a Diffeomorphism on a Phase Flow	75
§ 6. Symmetries	76
1. Symmetry Groups	76
2. Application of a One-parameter Symmetry Group to Integrate an Equation	77
3. Homogeneous Equations	79
4. Quasi-homogeneous Equations	82
5. Similarity and Dimensional Considerations	84
6. Methods of Integrating Differential Equations	86
Chapter 2. Basic Theorems	89
§ 7. Rectification Theorems	89
1. Rectification of a Direction Field	89
2. Existence and Uniqueness Theorems	92
3. Theorems on Continuous and Differentiable Dependence of the Solutions on the Initial Condition	93
4. Transformation over the Time Interval from t_0 to t	96
5. Theorems on Continuous and Differentiable Dependence on a Parameter	97
6. Extension Theorems	100
7. Rectification of a Vector Field	103
§ 8. Applications to Equations of Higher Order than First	104
1. The Equivalence of an Equation of Order n and a System of n First-order Equations	104
2. Existence and Uniqueness Theorems	107
3. Differentiability and Extension Theorems	108

4. Systems of Equations	109
5. Remarks on Terminology	112
§ 9. The Phase Curves of an Autonomous System	116
1. Autonomous Systems	117
2. Translation over Time	117
3. Closed Phase Curves	119
§ 10. The Derivative in the Direction of a Vector Field and First Integrals	121
1. The Derivative in the Direction of a Vector	121
2. The Derivative in the Direction of a Vector Field	122
3. Properties of the Directional Derivative	123
4. The Lie Algebra of Vector Fields	124
5. First Integrals	125
6. Local First Integrals	126
7. Time-Dependent First Integrals	127
§ 11. First-order Linear and Quasi-linear Partial Differential Equations	129
1. The Homogeneous Linear Equation	129
2. The Cauchy Problem	130
3. The Inhomogeneous Linear Equation	131
4. The Quasi-linear Equation	132
5. The Characteristics of a Quasi-linear Equation	133
6. Integration of a Quasi-linear Equation	135
7. The First-order Nonlinear Partial Differential Equation	136
§ 12. The Conservative System with one Degree of Freedom..	138
1. Definitions	138
2. The Law of Conservation of Energy	139
3. The Level Lines of the Energy	140
4. The Level Lines of the Energy Near a Singular Point	142
5. Extension of the Solutions of Newton's Equation	144
6. Noncritical Level Lines of the Energy	145
7. Proof of the Theorem of Sect. 6	146
8. Critical Level Lines	147
9. An Example	148
10. Small Perturbations of a Conservative System	149
Chapter 3. Linear Systems	152
§ 13. Linear Problems	152
1. Example: Linearization	152
2. Example: One-parameter Groups of Linear Transformations of R^n	153
3. The Linear Equation	154
§ 14. The Exponential Function	155

1. The Norm of an Operator	155
2. The Metric Space of Operators	156
3. Proof of Completeness	156
4. Series	157
5. Definition of the Exponential e^A	158
6. An Example	159
7. The Exponential of a Diagonal Operator	160
8. The Exponential of a Nilpotent Operator	160
9. Quasi-polynomials	161
§ 15. Properties of the Exponential	162
1. The Group Property	163
2. The Fundamental Theorem of the Theory of Linear Equations with Constant Coefficients	164
3. The General Form of One-parameter Groups of Linear Transformations of the Space R^n	165
4. A Second Definition of the Exponential	165
5. An Example: Euler's Formula for e^z	166
6. Euler's Broken Lines	167
§ 16. The Determinant of an Exponential	169
1. The Determinant of an Operator	169
2. The Trace of an Operator	170
3. The Connection Between the Determinant and the Trace	171
4. The Determinant of the Operator e^A	171
§ 17. Practical Computation of the Matrix of an Exponential – The Case when the Eigenvalues are Real and Distinct ..	173
1. The Diagonalizable Operator	173
2. An Example	174
3. The Discrete Case	175
§ 18. Complexification and Realification	177
1. Realification	177
2. Complexification	177
3. The Complex Conjugate	178
4. The Exponential, Determinant, and Trace of a Complex Operator ..	179
5. The Derivative of a Curve with Complex Values	180
§ 19. The Linear Equation with a Complex Phase Space	181
1. Definitions	181
2. The Fundamental Theorem	181
3. The Diagonalizable Case	182
4. Example: A Linear Equation whose Phase Space is a Complex Line	182
5. Corollary	185
§ 20. The Complexification of a Real Linear Equation	185
1. The Complexified Equation	185

2. The Invariant Subspaces of a Real Operator.....	187
3. The Linear Equation on the Plane.....	189
4. The Classification of Singular Points in the Plane.....	190
5. Example: The Pendulum with Friction.....	191
6. The General Solution of a Linear Equation in the Case when the Characteristic Equation Has Only Simple Roots.....	193
§ 21. The Classification of Singular Points of Linear Systems .	195
1. Example: Singular Points in Three-dimensional Space.....	195
2. Linear, Differentiable, and Topological Equivalence.....	197
3. The Linear Classification	198
4. The Differentiable Classification	199
§ 22. The Topological Classification of Singular Points.....	199
1. Theorem	199
2. Reduction to the Case $m_- = 0$	200
3. The Lyapunov Function	201
4. Construction of the Lyapunov Function	202
5. An Estimate of the Derivative	204
6. Construction of the Homeomorphism h	206
7. Proof of Lemma 3	207
8. Proof of the Topological Classification Theorem.....	208
§ 23. Stability of Equilibrium Positions.....	210
1. Lyapunov Stability	210
2. Asymptotic Stability	211
3. A Theorem on Stability in First Approximation.....	211
4. Proof of the Theorem.....	212
§ 24. The Case of Purely Imaginary Eigenvalues	215
1. The Topological Classification	215
2. An Example.....	215
3. The Phase Curves of Eq. (4) on the Torus	217
4. Corollaries	219
5. The Multidimensional Case.....	219
6. The Uniform Distribution	220
§ 25. The Case of Multiple Eigenvalues.....	221
1. The Computation of e^{At} , where A is a Jordan Block	221
2. Applications.....	223
3. Applications to Systems of Equations of Order Higher than the First	224
4. The Case of a Single n th-order Equation.....	225
5. On Recursive Sequences	226
6. Small Oscillations	227
§ 26. Quasi-polynomials.....	229
1. A Linear Function Space	229
2. The Vector Space of Solutions of a Linear Equation.....	230

3. Translation-invariance	231
4. Historical Remark	232
5. Inhomogeneous Equations	233
6. The Method of Complex Amplitudes	235
7. Application to the Calculation of Weakly Nonlinear Oscillations	240
§ 27. Nonautonomous Linear Equations	241
1. Definition	241
2. The Existence of Solutions	242
3. The Vector Space of Solutions	244
4. The Wronskian Determinant	245
5. The Case of a Single Equation	246
6. Liouville's Theorem	248
7. Sturm's Theorems on the Zeros of Solutions of Second-order Equations	251
§ 28. Linear Equations with Periodic Coefficients	256
1. The Mapping over a Period	256
2. Stability Conditions	258
3. Strongly Stable Systems	259
4. Computations	262
§ 29. Variation of Constants	264
1. The Simplest Case	264
2. The General Case	264
3. Computations	265
Chapter 4. Proofs of the Main Theorems	267
§ 30. Contraction Mappings	267
1. Definition	267
2. The Contraction Mapping Theorem	268
3. Remark	269
§ 31. Proof of the Theorems on Existence and Continuous Dependence on the Initial Conditions	269
1. The Successive Approximations of Picard	269
2. Preliminary Estimates	271
3. The Lipschitz Condition	272
4. Differentiability and the Lipschitz Condition	272
5. The Quantities C, L, a', b'	273
6. The Metric Space M	274
7. The Contraction Mapping $A : M \rightarrow M$	275
8. The Existence and Uniqueness Theorem	276
9. Other Applications of Contraction Mappings	277
§ 32. The Theorem on Differentiability	279
1. The Equation of Variations	279
2. The Differentiability Theorem	280

3. Higher Derivatives with Respect to x	281
4. Derivatives in x and t	281
5. The Rectification Theorem	282
6. The Last Derivative	285
Chapter 5. Differential Equations on Manifolds	288
§ 33. Differentiable Manifolds	288
1. Examples of Manifolds	288
2. Definitions	288
3. Examples of Atlases	291
4. Compactness	293
5. Connectedness and Dimension	293
6. Differentiable Mappings	294
7. Remark	296
8. Submanifolds	296
9. An Example	297
§ 34. The Tangent Bundle. Vector Fields on a Manifold	298
1. The Tangent Space	298
2. The Tangent Bundle	299
3. A Remark on Parallelizability	301
4. The Tangent Mapping	302
5. Vector Fields	303
§ 35. The Phase Flow Defined by a Vector Field	304
1. Theorem	304
2. Construction of the Diffeomorphisms g^t for Small t	305
3. The Construction of g^t for any t	306
4. A Remark	307
§ 36. The Indices of the Singular Points of a Vector Field	309
1. The Index of a Curve	309
2. Properties of the Index	310
3. Examples	310
4. The Index of a Singular Point of a Vector Field	312
5. The Theorem on the Sum of the Indices	313
6. The Sum of the Indices of the Singular Points on a Sphere	315
7. Justification	317
8. The Multidimensional Case	318
Examination Topics	323
Sample Examination Problems	324
Supplementary Problems	326
Subject Index	331