

Michael Molloy
Bruce Reed

Graph Colouring and the Probabilistic Method

With 19 Figures



Springer

Contents

Part I. Preliminaries

1. Colouring Preliminaries	3
1.1 The Basic Definitions	3
1.2 Some Classical Results	5
1.3 Fundamental Open Problems	7
1.4 A Point of View	9
1.5 A Useful Technical Lemma	10
1.6 Constrained Colourings and the List Chromatic Number	11
1.7 Intelligent Greedy Colouring	12
Exercises	13
2. Probabilistic Preliminaries	15
2.1 Finite Probability Spaces	15
2.2 Random Variables and Their Expectations	17
2.3 One Last Definition	19
2.4 The Method of Deferred Decisions	20
Exercises	21

Part II. Basic Probabilistic Tools

3. The First Moment Method	27
3.1 2-Colouring Hypergraphs	28
3.2 Triangle-Free Graphs with High Chromatic Number	29
3.3 Bounding the List Chromatic Number as a Function of the Colouring Number	31
3.3.1 An Open Problem	33
3.4 The Cochromatic Number	34
Exercises	36

4. The Lovász Local Lemma	39
4.1 Constrained Colourings	
and the List Chromatic Number	41
Exercises	42
5. The Chernoff Bound	43
5.1 Hajós's Conjecture	44
Exercises	46

Part III. Vertex Partitions

6. Hadwiger's Conjecture	49
6.1 Step 1: Finding a Dense Subgraph	50
6.2 Step 2: Finding a Split Minor	50
6.3 Step 3: Finding the Minor	52
Exercises	53
7. A First Glimpse of Total Colouring	55
8. The Strong Chromatic Number	61
Exercises	65
9. Total Colouring Revisited	67
9.1 The Idea	67
9.2 Some Details	70
9.3 The Main Proof	74
Exercises	75

Part IV. A Naive Colouring Procedure

10. Talagrand's Inequality	
and Colouring Sparse Graphs	79
10.1 Talagrand's Inequality	79
10.2 Colouring Triangle-Free Graphs	83
10.3 Colouring Sparse Graphs	86
10.4 Strong Edge Colourings	87
Exercises	89
11. Azuma's Inequality and a Strengthening	
of Brooks' Theorem	91
11.1 Azuma's Inequality	91
11.2 A Strengthening of Brooks' Theorem	94
11.3 The Probabilistic Analysis	98

11.4 Constructing the Decomposition	100
Exercises	103

Part V. An Iterative Approach

12. Graphs with Girth at Least Five	107
12.1 Introduction	107
12.2 A Wasteful Colouring Procedure	109
12.2.1 The Heart of The Procedure	109
12.2.2 The Finishing Blow	111
12.3 The Main Steps of the Proof	112
12.4 Most of the Details	115
12.5 The Concentration Details	120
Exercises	123
13. Triangle-Free Graphs	125
13.1 An Outline	126
13.1.1 A Modified Procedure	126
13.1.2 Fluctuating Probabilities	128
13.1.3 A Technical Fiddle	130
13.1.4 A Complication	131
13.2 The Procedure	131
13.2.1 Dealing with Large Probabilities	131
13.2.2 The Main Procedure	132
13.2.3 The Final Step	132
13.2.4 The Parameters	133
13.3 Expectation and Concentration	136
Exercises	138
14. The List Colouring Conjecture	139
14.1 A Proof Sketch	140
14.1.1 Preliminaries	140
14.1.2 The Local Structure	140
14.1.3 Rates of Change	141
14.1.4 The Preprocessing Step	142
14.2 Choosing $Reserve_e$	144
14.3 The Expected Value Details	145
14.4 The Concentration Details	149
14.5 The Wrapup	151
14.6 Linear Hypergraphs	152
Exercises	153

Part VI. A Structural Decomposition

15. The Structural Decomposition	157
15.1 Preliminary Remarks	157
15.2 The Decomposition	157
15.3 Partitioning the Dense Sets	160
15.4 Graphs with χ Near Δ	165
15.4.1 Generalizing Brooks' Theorem	165
15.4.2 Blowing Up a Vertex	166
Exercises	167
16. ω, Δ and χ	169
16.1 The Modified Colouring Procedure	171
16.2 An Extension of Talagrand's Inequality	172
16.3 Strongly Non-Adjacent Vertices	173
16.4 Many Repeated Colours	175
16.5 The Proof of Theorem 16.5	179
16.6 Proving the Harder Theorems	181
16.7 Two Proofs	182
Exercises	184
17. Near Optimal Total Colouring I: Sparse Graphs	185
17.1 Introduction	185
17.2 The Procedure	187
17.3 The Analysis of the Procedure	188
17.4 The Final Phase	191
18. Near Optimal Total Colouring II: General Graphs	195
18.1 Introduction	195
18.2 Phase I: An Initial Colouring	198
18.2.1 Ornerly Sets	198
18.2.2 The Output of Phase I	200
18.2.3 A Proof Sketch	201
18.3 Phase II: Colouring the Dense Sets	206
18.3.1 $\mathcal{T}_i$ is Non-Empty	207
18.3.2 Our Distribution is Nearly Uniform	208
18.3.3 Completing the Proof	209
18.4 Phase III: The Temporary Colours	210
18.4.1 Step 1: The Kernels of the Ornerly Sets	211
18.4.2 Step 2: The Remaining Temporary Colours	215
18.5 Phase IV – Finishing the Sparse Vertices	216
18.6 The Ornerly Set Lemmas	217

Part VII. Sharpening our Tools

19. Generalizations of the Local Lemma	221
19.1 Non-Uniform Hypergraph Colouring	222
19.2 More Frugal Colouring	224
19.2.1 Acyclic Edge Colouring	225
19.3 Proofs	226
19.4 The Lopsided Local Lemma	228
Exercises	229
20. A Closer Look at Talagrand's Inequality	231
20.1 The Original Inequality	231
20.2 More Versions	234
Exercises	236

Part VIII. Colour Assignment via Fractional Colouring

21. Finding Fractional Colourings and Large Stable Sets	239
21.1 Fractional Colouring	239
21.2 Finding Large Stable Sets in Triangle-Free Graphs	242
21.3 Fractionally, $\chi \leq \frac{\omega + \Delta + 1}{2}$	244
Exercises	246
22. Hard-Core Distributions on Matchings	247
22.1 Hard-Core Distributions	247
22.2 Hard-Core Distributions from Fractional Colourings	249
22.3 The Mating Map	252
22.4 An Independence Result	254
22.5 More Independence Results	260
23. The Asymptotics of Edge Colouring Multigraphs	265
23.1 Assigning the Colours	265
23.1.1 Hard-Core Distributions and Approximate Independence	266
23.2 The Chromatic Index	267
23.3 The List Chromatic Index	270
23.3.1 Analyzing an Iteration	272
23.3.2 Analyzing a Different Procedure	274
23.3.3 One More Tool	277
23.4 Comparing the Procedures	279
23.4.1 Proving Lemma 23.9	282

Part IX. Algorithmic Aspects

24. The Method of Conditional Expectations	287
24.1 The Basic Ideas	287
24.2 An Algorithm	288
24.3 Generalized Tic-Tac-Toe	289
24.4 Proof of Lemma 24.3	291
25. Algorithmic Aspects of the Local Lemma	295
25.1 The Algorithm	296
25.1.1 The Basics	296
25.1.2 Further Details	299
25.2 A Different Approach	300
25.3 Applicability of the Technique	301
25.3.1 Further Extensions	303
25.4 Extending the Approach	304
25.4.1 3-Uniform Hypergraphs	305
25.4.2 k -Uniform Hypergraphs with $k \geq 4$	308
25.4.3 The General Technique	310
Exercises	312
References	314
Index	323