

Contents

Preface — VII

1	Riemannian and sub-Riemannian geometries — 1
1.1	(X, G) -structures on manifolds — 1
1.2	Structure developments and holonomy homomorphisms — 4
1.3	Conformal geometry on the sphere — 7
1.3.1	Spaces of constant positive curvature — 7
1.3.2	Quasiconformal mappings — 8
1.3.3	Quasiregular mappings — 12
1.3.4	Conformal structures — 15
1.3.5	Isometric spheres of Möbius transformations — 17
1.4	The real hyperbolic space $\mathbb{H}_{\mathbb{R}}^n$ — 19
1.4.1	Hyperboloid model — 20
1.4.2	Projective (Klein) model — 21
1.4.3	Hemisphere model — 21
1.4.4	Poincaré ball and half-space (conformal) models of $\mathbb{H}_{\mathbb{R}}^n$ — 22
1.4.5	Busemann function and horospheres in $\mathbb{H}_{\mathbb{R}}^n$ — 24
1.4.6	Examples of real hyperbolic manifolds — 26
1.5	Lie subgroups of the Möbius group — 28
1.6	Geometry of orbifolds — 32
1.7	Hyperbolic geometries with Hermitian structures — 37
1.7.1	$\mathbb{F}$ -hyperbolic spaces — 37
1.7.2	Spherical CR-structures — 40
1.7.3	Ball model of the complex hyperbolic space — 41
1.7.4	Siegel domain: halfspace model of complex hyperbolic space — 42
1.7.5	Models of quaternionic hyperbolic space — 44
1.7.6	Cayley hyperbolic plane — 45
1.8	Nilpotent geometries on horospheres in negatively curved spaces — 46
1.8.1	Horospheres in $\mathbb{F}$ -hyperbolic spaces — 47
1.8.2	Chains in nilpotent geometry and inversions — 49
1.8.3	Geometry of Heisenberg group $\mathcal{H}_{2n-1}$ — 50
1.9	Basic manifolds topology for discrete group action — 53
1.9.1	Connected sum of manifolds — 53
1.9.2	3-manifolds topology — 53
1.9.3	Dehn surgery — 57
1.9.4	(X, G) pseudomanifolds made by gluing polyhedra faces — 58
	Notes — 59
2	Dynamics of discontinuous groups of homeomorphisms — 61
2.1	Convergence groups — 61

2.1.1	Compactness principle —	61
2.1.2	Elementary discrete convergence groups —	65
2.1.3	The limit set of a convergence group —	68
2.2	Group action on the discontinuity set —	72
2.2.1	The quotient space —	72
2.2.2	Basic classes of discontinuous groups —	75
2.2.3	Connected components of the discontinuity set —	80
2.3	Fundamental domains —	83
2.3.1	Fundamental polyhedra —	83
2.3.2	Ford and isometric polyhedra —	88
2.3.3	Bisectors in symmetric spaces and Dirichlet polyhedra —	90
2.3.4	Pseudo-Dirichlet polyhedra —	97
2.3.5	Polyhedral representations of groups —	101
2.4	Convex polyhedra and reflection groups —	104
2.4.1	Coxeter diagrams and Gram matrices —	104
2.4.2	Andreev rigidity and classification of reflection groups —	107
2.4.3	Cofinite reflection groups in low dimensions —	111
2.5	Dynamics of discrete group action on the limit set —	113
2.5.1	Infinite solid angle of G -orbit and ergodicity on $S^n \times S^n$ —	113
2.5.2	Conical limit points and ergodicity on $S^n \times S^n$ —	117
2.5.3	Horospherical limit points and recurrence —	122
2.5.4	Conformal dynamical systems and Sullivan's microscope —	124
	Notes —	127
3	Basics of $\mathbb{H}$-hyperbolic lattices and manifolds —	128
3.1	Margulis lemma and thin-thick splittings of hyperbolic manifolds —	128
3.2	Dynamics of discrete group action on nilpotent Lie groups and Margulis region —	134
3.2.1	Extension of Bieberbach–Auslander theorems —	134
3.2.2	Group cohomology and group extensions —	135
3.2.3	Proof of structural Theorem 3.8 —	137
3.2.4	Margulis region and discrete action near parabolic fixed points —	141
3.3	Hyperbolic thin cusp submanifolds —	144
3.3.1	$\Gamma \subset \text{Isom } \mathbb{R}^{n-1}$ consists of pure translations —	145
3.3.2	$\Gamma \subset \text{Isom } \mathbb{R}^{n-1}$ does not have irrational screw translations —	146
3.3.3	$\Gamma \subset \text{Isom } \mathbb{R}^{n-1}$ has irrational screw translations —	146
3.3.4	Thin ends of oriented 3-manifolds —	147
3.3.5	Thin ends of n -manifolds, $n \geq 4$ —	148
3.4	Parabolic cusps and precisely invariant horoballs in symmetric rank one spaces —	151
3.4.1	Precisely invariant horoballs —	151
3.4.2	Group action on the set of horoballs —	155

3.4.3	Parabolic cusp points in symmetric rank 1 spaces —	158
3.5	Hyperbolic convex cores —	160
3.6	Hyperbolic arithmetics —	165
3.7	Arithmetic lattices generated by reflections —	169
3.7.1	Vinberg's criterion for real arithmeticity —	169
3.7.2	Makarov nonarithmetic lattices —	171
3.7.3	Nikulin–Vinberg–Prokhorov finiteness for reflection lattices —	172
3.7.4	Applications to algebraic geometry —	175
3.8	Gromov and Piatetski-Shapiro hyperbolic interbreeding —	176
3.8.1	Nonarithmetic lattices in all dimensions $n \geq 2$ —	176
3.8.2	Interbreeding hyperbolic manifolds —	178
3.8.3	Commensurability condition —	179
3.9	Fibonacci groups and manifolds —	180
3.10	Geometry of tessellations and hyperbolic lattices —	187
3.10.1	Aleksandrov's complex and space fillings —	188
3.10.2	Incomplete hyperbolic structures and tessellations —	190
3.10.3	Thurston nonrigidity of hyperbolic nonuniform lattices —	196
3.10.4	Poincaré–Aleksandrov theorem on fundamental polyhedron —	197
3.10.5	Rigidity of high-dimensional hyperbolic tessellations —	201
	Notes —	203

4	Geometrically finite actions in symmetric rank 1 spaces —	206
4.1	Classical finiteness for Kleinian groups —	206
4.1.1	Finiteness for planar Kleinian groups —	206
4.1.2	Geometrical finiteness and hyperbolic Dirichlet polyhedra —	210
4.2	The Beardon–Maskit finiteness condition on dynamics of discrete groups —	213
4.2.1	Conical limit points and parabolic cusps —	213
4.2.2	Finite-sided pseudo-Dirichlet polyhedra —	214
4.2.3	The limit set measure —	217
4.3	Definitions of geometrical finiteness in symmetric spaces —	219
4.3.1	Standard cusp ends —	219
4.3.2	Compactness of the thick part of the convex core —	220
4.3.3	Finite volume of the convex core —	221
4.3.4	Geometric finiteness conditions —	222
4.3.5	Geometrically finite ends and coverings —	227
4.4	Cayley graphs and geometric isomorphisms of discrete groups —	230
4.4.1	Cayley graphs and group completions —	230
4.4.2	Cayley graphs of hyperbolic isometry groups —	234
4.4.3	Group completion and the limit set —	236
4.4.4	Cayley graphs and geometric isomorphisms of discrete groups —	239
4.5	Geometrical finiteness for discontinuity set components —	243

4.5.1	Hermitian $\mathbb{F}$ -hyperbolic Kleinian manifolds —	243
4.5.2	Real hyperbolic Kleinian manifolds —	252
4.5.3	Connected components with cocompact G -action —	252
4.5.4	General case: discrete groups with parabolic elements —	255
4.6	Topology and finiteness problems for Kleinian manifolds —	263
4.6.1	Constructions of Kleinian manifolds —	263
4.6.2	Topology of Kleinian n -manifolds —	273
4.6.3	Finiteness problems for Kleinian n -manifolds —	278
	Notes —	286
5	Deformations of locally symmetric spaces —	289
5.1	Deformations of geometric structures —	289
5.1.1	Teichmüller spaces of marked structures —	289
5.1.2	Teichmüller space of Riemann surfaces —	289
5.1.3	Variety of group representations and stability —	291
5.1.4	Holonomy theorem and Sullivan structural stability —	293
5.1.5	Tangent bundle to Teichmüller space and group cohomology —	297
5.1.6	Tangent bundle to the representation variety $\text{Hom}(\Gamma, G)$ —	298
5.2	Rigidity of locally symmetric rank 1 structures —	303
5.2.1	Lattices and spaces of finite $\mathbb{F}$ -hyperbolic volume —	303
5.2.2	Hyperbolic structures of infinite volume —	305
5.2.3	Conformal rigidity of manifolds with boundary —	318
5.2.4	Weak rigidity of hyperbolic structures of infinite volume —	320
5.3	Bending deformations of locally symmetric structures —	324
5.3.1	Bendings of real structures along totally geodesic hypersurfaces —	324
5.3.2	Bendings of Hermitian structures —	333
5.3.3	Bending of a surface group in $\text{Isom } \mathbb{H}_{\mathbb{F}}^n$ —	337
5.3.4	Infinitesimal bendings and singularities of Teichmüller space $\mathcal{T}(M)$ —	340
5.4	Quasiconformally instable deformations of complex disc bundles —	343
5.5	Quasi-Fuchsian structures: cone deformations —	352
5.5.1	Cone singularities of hyperbolic convex hulls —	353
5.5.2	Polyhedral cone deformations and singularities of convex hulls —	357
5.5.3	Round cone deformations of Jørgensen–Thurston 3-manifolds —	362
5.6	Hinge-bendings along hypersurfaces with boundaries —	368
5.6.1	Hinge-bending along hinge-collections of geodesic hypercells —	368
5.6.2	Hinge-bendings of 3-manifolds along arbitrarily many halvesurfaces —	372
5.6.3	Hinge-bendings as bendings along pleated surfaces —	374
	Notes —	375
6	Hyperbolic 4-cobordisms and wild dynamics —	377
6.1	Kleinian manifolds and hyperbolic cobordisms —	377
6.1.1	Nontrivial hyperbolic homology 4-cobordisms —	378

6.1.2	Block-building construction of “wild” groups —	379
6.1.3	Quasisymmetric, everywhere wild Fox–Artin spheres —	387
6.1.4	Bendings and quasiconformal dynamics of “wild” group —	388
6.2	Representation varieties and ergodic dynamics of hyperbolic lattices —	394
6.2.1	Components of Teichmüller space and wild dynamics on knots —	395
6.2.2	Block-building construction for wild $(n - 2)$ -knots in the n -sphere —	399
6.2.3	Quasisymmetric conjugation of ergodic actions —	409
6.3	Symmetric hyperbolic cobordisms —	410
6.3.1	Teichmüller spaces, reflection groups, and cobordisms —	411
6.3.2	Block-building construction of symmetric 4-cobordism —	412
6.3.3	Nonfaithful discrete representations and symmetric cobordisms —	416
6.3.4	Discrete representations with large kernels —	418
6.4	Symmetric cobordisms and quasiregular mappings —	421
6.4.1	Dynamics of hyperbolic lattice action and equivariant mappings —	421
6.4.2	QC dynamics and topological barriers for QR mappings —	426
6.5	Conformal interbreeding and symmetric cobordisms —	427
6.5.1	Interbreeding of hyperbolic lattices and conformal interbreeding —	428
6.5.2	Conformal interbreeding and discrete representations of hyperbolic lattices with arbitrary large kernels —	428
6.5.3	QC dynamics and injectivity of bounded quasiregular mappings —	433
6.5.4	Asymptotics of bounded quasiregular mappings in the unit ball and growth in free groups —	435
6.5.5	Nontrivial hyperbolic homology 4-cobordisms and wild QC-dynamics —	436
	Notes —	444

Bibliography — 447

Index — 503