

Contents

1	Introduction	1
1.1	Dynamical Systems and Hyperbolicity	1
1.2	Ergodic Theory and Nontrivial Recurrence	2
1.3	Hyperbolicity and Dimension Theory	3
1.4	Geometric Constructions and Limit Sets	5
1.5	Classical Thermodynamic Formalism	6
1.6	Dimension Theory in Hyperbolic Dynamics	8
1.7	Multifractal Analysis and Irregular Sets	12
1.8	Quantitative Recurrence and Dimension	15

Part I Ergodic Theory

2	Basic Notions and Examples	19
2.1	Introduction	19
2.2	Invariant Measures	21
2.2.1	The Notion of Invariant Measure	21
2.2.2	Rotations and Translations of $\mathbb{R}^n$	23
2.2.3	Circle Rotations and Interval Translations	24
2.2.4	Expanding Maps of the Circle	27
2.2.5	Toral Endomorphisms	28
2.2.6	Piecewise Linear Expanding Maps	31
2.3	Poincaré's Recurrence Theorem	33
2.4	Invariant Sets and Invariant Functions	36
2.5	Birkhoff's Ergodic Theorem	37
2.5.1	Formulation of the Theorem	37
2.5.2	Conditional Expectation	39
2.5.3	Proof of Birkhoff's Ergodic Theorem	40
2.6	Ergodicity	43
2.6.1	Basic Notions	43
2.6.2	Fourier Coefficients and Ergodicity	45
2.6.3	The Case of Invariant Measures	48

2.7	Applications to Number Theory	48
2.7.1	Fractional Parts of Polynomials	48
2.7.2	Continued Fractions	51
2.8	Exercises	58
3	Further Topics	65
3.1	Introduction	65
3.2	Existence of Invariant Measures	66
3.2.1	Preliminary Results	67
3.2.2	Existence of Invariant Measures	71
3.3	Unique Ergodicity	72
3.3.1	Basic Notions and Uniform Convergence	72
3.3.2	Criteria for Unique Ergodicity and Examples	74
3.4	Mixing	80
3.5	Symbolic Dynamics	84
3.5.1	Basic Notions	85
3.5.2	Markov Measures and Bernoulli Measures	87
3.5.3	Topological Markov Chains and Markov Measures	91
3.5.4	The Case of Two-Sided Sequences	92
3.6	Topological Dynamics	94
3.7	Exercises	96

Part II Entropy and Pressure

4	Metric Entropy and Topological Entropy	107
4.1	Introduction	107
4.2	Metric Entropy	109
4.2.1	The Notion of Metric Entropy	109
4.2.2	Conditional Entropy	115
4.2.3	Generators and Examples	121
4.3	Shannon–McMillan–Breiman Theorem	125
4.4	Topological Entropy	132
4.4.1	Basic Notions and Some Properties	132
4.4.2	Topological Nature of the Entropy	137
4.5	Variational Principle	138
4.6	Exercises	143
5	Thermodynamic Formalism	147
5.1	Introduction	147
5.2	Topological Pressure	148
5.3	Symbolic Dynamics	149
5.4	Variational Principle	153
5.5	Equilibrium Measures	158
5.6	Exercises	164

Part III Hyperbolic Dynamics

6 Basic Notions and Examples	171
6.1 Introduction	171
6.2 Hyperbolic Sets	173
6.2.1 The Notion of Hyperbolicity	173
6.2.2 Some Properties of Hyperbolic Sets	176
6.3 Smale Horseshoe	178
6.3.1 Construction of the Horseshoe	178
6.3.2 Symbolic Dynamics	182
6.4 Hyperbolic Automorphisms of $\mathbb{T}^2$	185
6.4.1 Construction of a Coding Map	185
6.4.2 Induction of a Markov Measure	190
6.5 The Case of Repellers	193
6.6 Exercises	197
7 Invariant Manifolds and Markov Partitions	201
7.1 Introduction	201
7.2 Stable and Unstable Manifolds	202
7.3 Ergodicity and Hopf's Argument	211
7.4 Product Structure	215
7.5 The Shadowing Property	217
7.6 Construction of Markov Partitions	219
7.7 Exercises	228

Part IV Dimension Theory

8 Basic Notions and Examples	235
8.1 Introduction	235
8.2 Dimension of Sets	236
8.2.1 Basic Notions	236
8.2.2 Examples	237
8.3 Dimension of Measures	242
8.3.1 Basic Notions and Examples	242
8.3.2 Dimension Estimates via Pointwise Dimension	244
8.4 Exercises	249
9 Dimension Theory of Hyperbolic Dynamics	253
9.1 Introduction	253
9.2 Repellers	254
9.2.1 Conformal Maps and Examples	255
9.2.2 Dimension of Repellers	256
9.3 Hyperbolic Sets	265
9.3.1 Dimension Along the Invariant Manifolds	265
9.3.2 Dimension of Hyperbolic Sets	271
9.4 Exercises	272

A	Notions from Measure Theory	277
A.1	Measure Spaces	277
A.2	Outer Measures and Measurable Sets	278
A.3	Measures in Topological Spaces	278
A.4	Measurable Functions, Integration, and Convergence	279
A.5	Absolutely Continuous Measures	281
A.6	Product Spaces	282
	References	283
	Index	287