Contents

Part I Curves

1 Affine and Projective Space 3
1.1 Definitions 3
1.2 Algebraic Subsets and Coordinates 5
1.3 Affine and Projective Coordinate Systems 6
1.4 The Theorem of Desargues 8
1.5 Duality for $\mathbb{P}^2_k$ 11

2 Curves in $\mathbb{A}^2_k$ and in $\mathbb{P}^2_k$ 13
2.1 Conic Sections 13
2.2 Singular and Non-singular Points 14
2.3 Conics in the Projective Plane 17
2.4 The Cubic Curves in $\mathbb{A}^2_k$ 20
2.5 Elliptic Integrals and the Elliptic Transcendentals 24
2.6 More Curves in $\mathbb{A}^2_{\mathbb{R}}$ 26
2.7 General Affine Algebraic Curves 29
2.8 Singularities and Multiplicities 32
2.9 Tangency 35

3 Higher Geometry in the Projective Plane 39
3.1 Projective Curves 39
3.2 Projective Closure and Affine Restriction 41
3.3 Smooth and Singular Points on Affine and Projective Curves 43
3.4 The Tangent to a Projective Curve 46
3.5 Projective Equivalence 54
3.6 Asymptotes 55
3.7 General Conchoids 56
3.8 The Dual Curve 59

4 Plane Curves and Algebra 63
4.1 Affine and Homogeneous Coordinate Rings 63

Bibliografische Informationen
http://d-nb.info/1009641727

digitalisiert durch DEUTSCHE NATIONAL BIBLIOTHEK

4.2 Multiplicity and the Local Rings . . . 69
4.3 Intersection Multiplicities for Affine Plane Curves . . . 74
4.4 Intersection Theory for Curves in $\mathbb{P}^2_k$ . . . 76
4.5 Valuations and Valuation Rings . . . 85
4.6 Linear Systems of Plane Curves . . . 91
4.7 Affine Restriction and Projective Closure . . . 96
4.8 Bézout's Theorem . . . 97
4.9 Algebraic Derivatives and the Jacobian . . . 105
4.10 Simple Elimination Theory . . . 108
4.11 An Application: The Twisted Cubic Curve . . . 110
4.12 Points of Inflexion and the Hessian . . . 111

5 Projective Varieties in $\mathbb{P}^N_k$ . . . 119
5.1 Subvarieties of $\mathbb{P}^N_k$ . . . 119
5.2 Projective Non-singular Curves . . . 125
5.3 Divisors on a Projective Non-singular Curve . . . 126
5.4 Smoothness and Tangency in any Dimension . . . 129
5.5 Hilbert Polynomial and Projective Invariants . . . 129
5.6 Emmy Noether, Her Family and Their Fate . . . 135
5.7 The Riemann-Roch Theorem for Non-singular Curves . . . 136

Part II Introduction to Grothendieck's Theory of Schemes

6 Categories and Functors . . . 143
6.1 Objects and Morphisms . . . 143
6.2 Examples of Categories . . . 144
6.3 The Dual Category . . . 145
6.4 The Topology on a Topological Space Viewed as a Category 146
6.5 Monomorphisms and Epimorphisms . . . 146
6.6 Isomorphisms . . . 148
6.7 Covariant and Contravariant Functors . . . 148
6.8 Forgetful Functors . . . 150
6.9 The Category of Functors $\mathcal{F}un(\mathcal{C}, \mathcal{D})$ . . . 150
6.10 Functors of Several Variables . . . 150
6.11 Isomorphic and Equivalent Categories . . . 151
6.12 When are two Functors Isomorphic? . . . 154
6.13 Left and Right Adjoint Functors . . . 155
6.14 Representable Functors . . . 157
6.15 Representable Functors and Universal Properties: Yoneda's Lemma . . . 158

7 Constructions and Representable Functors . . . 161
7.1 Products and Coproducts . . . 161
7.2 Fibered Products and Coproducts . . . 163

8 Abelian Categories . . . 165
8.1 Definitions . . . 165

8.2 Product and Coproduct in the Non Abelian Category of Commutative Rings . . . 166
8.3 Localization as Representing a Functor . . . 166
8.4 Kernel and Cokernel of Two Morphisms . . . 166
8.5 Inductive and Projective Limits . . . 168
8.6 Important Special Case: Projective and Inductive Systems and Their Limits . . . 170
8.7 On the Existence of Projective and Inductive Limits . . . 171
8.8 The Stalk of a Presheaf on a Topological Space . . . 174
8.9 Grothendieck Topologies, Sheaves and Presheaves . . . 175
8.10 The Sheaf Associated to a Presheaf . . . 178
8.11 The Category of Abelian Sheaves . . . 180

9 The Concept of Spec(A) . . . 185
9.1 The Affine Spectrum of a Commutative Ring . . . 185
9.2 Very first Examples of Affine Spectra . . . 193

10 The Category of Schemes . . . 195
10.1 First Approximation: The Category of Ringed Spaces . . . 195
10.2 Second Approximation: Local Ringed Spaces . . . 198
10.3 Definition of the Category of Schemes . . . 204
10.4 Formal Properties of Products . . . 212

11 Properties of Morphisms of Schemes . . . 217
11.1 Modules and Algebras on Schemes . . . 217
11.2 Spec of an $\mathcal{O}_X$-Algebra on a Scheme X . . . 223
11.3 Reduced Schemes and the Reduced Subscheme X_{red} of X . . . 224
11.4 Reduced and Irreducible Schemes and the Field of Functions 225
11.5 Irreducible Components of Noetherian Schemes . . . 226
11.6 Embeddings, Graphs and the Diagonal . . . 227
11.7 Some Concepts from General Topology: A Reminder . . . 232
11.8 Separated Morphisms and Separated Schemes . . . 234

12 Modules, Algebras and Bundles on a Scheme . . . 239
12.1 The Category of $\mathcal{O}_X$-Modules on a Scheme X . . . 239
12.2 Linear Algebra on a Scheme . . . 240
12.3 Locally Free Modules on a Scheme . . . 241
12.4 Vector Bundles on a Scheme X . . . 244

13 More Properties of Morphisms, Scheme Theoretic Image and the "Sorite" . . . 247
13.1 Dominant Morphisms and Scheme Theoretic Image of a Subscheme . . . 247
13.2 Quasi Compact Morphisms . . . 248
13.3 Finiteness Conditions . . . 249
13.4 The Sorite for Properties of Morphisms . . . 250

14 Projective Schemes and Bundles 253
14.1 Algebraic Schemes over k and k-Varieties 253
14.2 Definition of Proj(S) as a Topological Space 255
14.3 The Spec(A)-Scheme Proj(S) and Its Sheaves of Modules . . 258
14.4 Proj of a Graded $\mathcal{O}_X$-Algebra on X 260
14.5 Projective Bundles and the Projective N-space over a Scheme 260

15 Further Properties of Morphisms 263
15.1 Affine and Projective Morphisms 263
15.2 Valuational Criteria for Separated and for Proper 265
15.3 Basic Properties of Proj($\mathcal{S}$) 267
15.4 Very Ample Sheaves 267
15.5 The Segre Embedding 269
15.6 Algebraic Complement 269
15.7 The Functorial Behavior of Proj 271
15.8 Tensor Products of Graded S-Modules 272
15.9 Basic Facts on Global Proj 273

16 Conormal Sheaf and Projective Bundles 275
16.1 The Conormal Sheaf 275
16.2 Kähler Differentials of an S-Scheme X 275
16.3 The Universal 1-Quotient on $\mathbb{P}(\mathcal{E})$ 279
16.4 The Segre Embedding for Projective Bundles 282
16.5 Base Extensions of Projective Morphisms 283
16.6 Regular Schemes 283

17 Cohomology Theory on Schemes 285
17.1 Some Homological Algebra 285
17.2 Derived Functors 292
17.3 Grothendieck Cohomology 294
17.4 Flasque, or Flabby Sheaves 296
17.5 Čech Cohomology 299

18 Intersection Theory 307
18.1 Basic Facts on Divisors 307
18.2 Chow Homology and Chow Cohomology 309

19 Characteristic Classes in Algebraic Geometry 321
19.1 Basics on $\mathbb{P}(\mathcal{E})$ Revisited 321
19.2 Chern Classes, Chern Characters and Todd Classes 323
19.3 Serre Duality 326

20 The Riemann-Roch Theorem 329
20.1 Hirzebruch's Riemann-Roch Theorem 329
20.2 The Riemann-Roch Theorem for Curves, Revisited 330
20.3 The Riemann-Roch Theorem for a Surface 330
20.4 Grothendieck's Riemann-Roch Theorem 333

21 Some Basic Constructions in the Category of Projective k-Varieties 335
21.1 The Blowing-up of a Closed Subscheme 335
21.2 Blowing Up of Subbundles 337
21.3 Grassmann Bundles 339
21.4 The Parameter Variety for the Joining Lines 339
21.5 The Secant Variety and the Join 341

22 More on Duality 343
22.1 The Dual Variety and the Conormal Scheme 343
22.2 Reflexivity and Biduality 346
22.3 Duality of Projective Varieties 347
22.4 Duality of Hyperplane Sections and Projections 350
22.5 A Theorem of Hefez-Kleiman 353

References 357

Index 359