

Contents

Exact solutions of nonlinear partial differential equations by singularity analysis

<i>Robert Conte</i>	1
1 Introduction	1
2 Various levels of integrability for PDEs, definitions	2
3 Importance of the singularities: a brief survey of the theory of Painlevé	9
4 The Painlevé test for PDEs in its invariant version	11
4.1 Singular manifold variable φ and expansion variable χ	11
4.2 The WTC part of the Painlevé test for PDEs	14
4.3 The various ways to pass or fail the Painlevé test for PDEs ..	17
5 Ingredients of the “singular manifold method”	18
5.1 The ODE situation	19
5.2 Transposition of the ODE situation to PDEs	19
5.3 The singular manifold method as a singular part transformation	20
5.4 The degenerate case of linearizable equations	21
5.5 Choices of Lax pairs and equivalent Riccati pseudopotentials. Second-order Lax pairs and their privilege	21
Third-order Lax pairs	23
5.6 The admissible relations between τ and ψ	24
6 The algorithm of the singular manifold method	24
6.1 Where to truncate, and with which variable?	27
7 The singular manifold method applied to one-family PDEs	29
7.1 Integrable equations with a second order Lax pair	29
The Liouville equation	30
The AKNS equation	32
The KdV equation	33
7.2 Integrable equations with a third order Lax pair	35
The Boussinesq equation	35
The Hirota-Satsuma equation	37
The Tzitzéica equation	38
The Sawada-Kotera and Kaup-Kupershmidt equations	43
The Sawada-Kotera equation	44

	The Kaup-Kupershmidt equation	45
7.3	Nonintegrable equations, second scattering order	49
	The Kuramoto-Sivashinsky equation	49
7.4	Nonintegrable equations, third scattering order	52
8	Two common errors in the one-family truncation	53
8.1	The constant level term does not define a BT	53
8.2	The WTC truncation is suitable iff the Lax order is two	54
9	The singular manifold method applied to two-family PDEs	54
9.1	Integrable equations with a second order Lax pair	55
	The sine-Gordon equation	55
	The modified Korteweg-de Vries equation	57
	The nonlinear Schrödinger equation	59
9.2	Integrable equations with a third order Lax pair	59
9.3	Nonintegrable equations, second and third scattering order	60
	The KPP equation	60
	The cubic complex Ginzburg-Landau equation	65
	The nonintegrable Kundu-Eckhaus equation	68
10	Singular manifold method <i>versus</i> reduction methods	69
11	Truncation of the unknown, not of the equation	72
12	Birational transformations of the Painlevé equations	74
13	Conclusion, open problems	76
	References	77

The method of Poisson pairs in the theory of nonlinear PDEs

	<i>Franco Magri, Gregorio Falqui, Marco Pedroni</i>	85
1	Introduction: The tensorial approach and the birth of the method of Poisson pairs	85
1.1	The Miura map and the KdV equation	86
1.2	Poisson pairs and the KdV hierarchy	88
1.3	Invariant submanifolds and reduced equations	90
1.4	The modified KdV hierarchy	94
2	The method of Poisson pairs	96
3	A first class of examples and the reduction technique	101
3.1	Lie-Poisson manifolds	101
3.2	Polynomial extensions	102
3.3	Geometric reduction	103
3.4	An explicit example	104
3.5	A more general example	108
4	The KdV theory revisited	109
4.1	Poisson pairs on a loop algebra	109
4.2	Poisson reduction	110
4.3	The GZ hierarchy	112
4.4	The central system	113
4.5	The linearization process	115
4.6	The relation with the Sato approach	117

5	Lax representation of the reduced KdV flows	120
5.1	Lax representation	120
5.2	First example	122
5.3	The generic stationary submanifold	124
5.4	What more?	125
6	Darboux–Nijenhuis coordinates and separability	125
6.1	The Poisson pair	126
6.2	Passing to a symplectic leaf	128
6.3	Darboux–Nijenhuis coordinates	130
6.4	Separation of variables	131
	References	134

Nonlinear superposition formulae of integrable partial differential equations by the singular manifold method

	<i>Micheline Musette</i>	137
1	Introduction	137
2	Integrability by the singularity approach	138
3	Bäcklund transformation: definition and example	139
4	Singularity analysis of nonlinear differential equations	139
4.1	Nonlinear ordinary differential equations	139
4.2	Nonlinear partial differential equations	142
5	Lax Pair and Darboux transformation	143
5.1	Second order scalar scattering problem	144
5.2	Third order scalar scattering problem	145
5.3	A third order matrix scattering problem	146
6	Different truncations in Painlevé analysis	147
7	Method for a one-family equation	149
8	Nonlinear superposition formula	151
9	Results for PDEs possessing a second order Lax pair	151
9.1	First example: KdV equation	151
9.2	Second example: MKdV and sine-Gordon equations	153
10	PDEs possessing a third order Lax pair	156
10.1	Sawada-Kotera, KdV ₅ , Kaup-Kupershmidt equations	156
10.2	Painlevé test	157
10.3	Truncation with a second order Lax pair	158
10.4	Truncation with a third order Lax pair	158
10.5	Bäcklund transformation	159
10.6	Nonlinear superposition formula for Sawada-Kotera	160
10.7	Nonlinear superposition formula for Kaup-Kupershmidt	161
10.8	Tzitzéica equation	165
	References	166

Hirota bilinear method for nonlinear evolution equations

<i>Junkichi Satsuma</i>	171
1 Introduction	171
2 Soliton solutions	172
2.1 The Burgers equation	172
2.2 The Korteweg-de Vries equation	173
2.3 The nonlinear Schrödinger equation	174
2.4 The Toda equation	175
2.5 Painlevé equations	176
2.6 Difference vs differential	177
3 Multidimensional equations	180
3.1 The Kadomtsev-Petviashvili equation	180
3.2 The two-dimensional Toda lattice equation	181
3.3 Two-dimensional Toda molecule equation	184
3.4 The Hirota-Miwa equation	185
4 Sato theory	187
4.1 Micro-differential operators	187
4.2 Introduction of an infinite number of time variables	189
4.3 The Sato equation	192
4.4 Generalized Lax equation	194
4.5 Structure of tau functions	195
4.6 Algebraic identities for tau functions	200
4.7 Vertex operators and the KP bilinear identity	204
4.8 Fermion analysis based on an infinite dimensional Lie algebra	207
5 Extensions of the bilinear method	210
5.1 q -discrete equations	210
5.2 Special function solution for soliton equations	212
5.3 Ultra discrete soliton system	215
5.4 Trilinear equations	218
References	221

Lie groups, singularities and solutions of nonlinear partial differential equations

<i>Pavel Winternitz</i>	223
1 Introduction	223
2 The symmetry group of a system of differential equations	225
2.1 Formulation of the problem	225
Prolongation	226
Symmetry group: Global approach, use the chain rule	227
Symmetry group: Infinitesimal approach	227
Reformulation	227
2.2 Prolongation of vector fields and the symmetry algorithm ..	228
2.3 First example: Variable coefficient KdV equation	230
2.4 Symmetry reduction for the KdV	232
2.5 Second example: Modified Kadomtsev-Petviashvili equation ..	235

3	Classification of the subalgebras of a finite dimensional Lie algebra	238
3.1	Formulation of the problem	238
3.2	Subalgebras of a simple Lie algebra	239
3.3	Example: Maximal subalgebras of $o(4, 2)$	240
3.4	Subalgebras of semidirect sums	244
3.5	Example: All subalgebras of $sl(3, \mathcal{R})$ classified under the group $SL(3, \mathcal{R})$	248
3.6	Generalizations	252
4	The Clarkson-Kruskal direct reduction method and conditional symmetries	252
4.1	Formulation of the problem	252
4.2	Symmetry reduction for Boussinesq equation	253
4.3	The direct method	254
4.4	Conditional symmetries	256
4.5	General comments	261
5	Concluding comments	263
5.1	References on nonlinear superposition formulas	263
5.2	References on continuous symmetries of difference equations	264
	References	264
	List of Participants	274
	Index	277