

Contents

Prologue.....	1
---------------	---

Part I. The Euclidean Manifold as a Paradigm

1. The Euclidean Space E_3 (Hilbert Space and Lie Algebra Structure)	69
1.1 A Glance at History.....	69
1.2 Algebraic Basic Ideas.....	71
1.2.1 Symmetrization and Antisymmetrization	72
1.2.2 Cramer's Rule for Systems of Linear Equations	73
1.2.3 Determinants and the Inverse Matrix	75
1.2.4 The Hilbert Space Structure	78
1.2.5 Orthogonality and the Dirac Calculus	81
1.2.6 The Lie Algebra Structure	82
1.2.7 The Metric Tensor	85
1.2.8 The Volume Form.....	85
1.2.9 Grassmann's Alternating Product	86
1.2.10 Perspectives	87
1.3 The Skew-Field \mathbb{H} of Quaternions	89
1.3.1 The Field \mathbb{C} of Complex Numbers	90
1.3.2 The Galois Group $\text{Gal}(\mathbb{C} \mathbb{R})$ and Galois Theory	91
1.3.3 A Glance at the History of Hamilton's Quaternions..	94
1.3.4 Pauli's Spin Matrices and the Lie Algebras $su(2)$ and $sl(2, \mathbb{C})$	99
1.3.5 Cayley's Matrix Approach to Quaternions.....	101
1.3.6 The Unit Sphere $U(1, \mathbb{H})$ and the Electroweak Gauge Group $SU(2)$	102
1.3.7 The Four-Dimensional Extension of the Euclidean Space E_3	103
1.3.8 Hamilton's Nabla Operator.....	104
1.3.9 The Indefinite Hilbert Space \mathbb{H} and the Minkowski Space M_4	104
1.4 Riesz Duality between Vectors and Covectors.....	104

XV

1.5	The Heisenberg Group, the Heisenberg Algebra, and Quantum Physics	106
1.6	The Heisenberg Group Bundle and Gauge Transformations ..	112
2.	Algebras and Duality (Tensor Algebra, Grassmann Algebra, Clifford Algebra, Lie Algebra)	115
2.1	Multilinear Functionals	115
2.1.1	The Graded Algebra of Polynomials	115
2.1.2	Products of Multilinear Functionals	118
2.1.3	Tensor Algebra	120
2.1.4	Grassmann Algebra (Alternating Algebra)	121
2.1.5	Symmetric Tensor Algebra	121
2.1.6	The Universal Property of the Tensor Product	122
2.1.7	Diagram Chasing	124
2.2	The Clifford Algebra $\bigvee(E_1)$ of the One-Dimensional Euclidean Space E_1	126
2.3	Algebras of the Two-Dimensional Euclidean Space E_2	127
2.3.1	The Clifford Algebra $\bigvee(E_2)$ and Quaternions	128
2.3.2	The Cauchy–Riemann Differential Equations in Complex Function Theory	129
2.3.3	The Grassmann Algebra $\bigwedge(E_2)$	131
2.3.4	The Grassmann Algebra $\bigwedge(E_2^d)$	132
2.3.5	The Symplectic Structure of E_2	132
2.3.6	The Tensor Algebra $\bigotimes(E_2)$	133
2.3.7	The Tensor Algebra $\bigotimes(E_2^d)$	133
2.4	Algebras of the Three-Dimensional Euclidean Space E_3	133
2.4.1	Lie Algebra	133
2.4.2	Tensor Algebra	133
2.4.3	Grassmann Algebra	134
2.4.4	Clifford Algebra	134
2.5	Algebras of the Dual Euclidean Space E_3^d	135
2.5.1	Tensor Algebra	135
2.5.2	Grassmann Algebra	135
2.6	The Mixed Tensor Algebra	136
2.7	The Hilbert Space Structure of the Grassmann Algebra (Hodge Duality)	138
2.7.1	The Hilbert Space $\bigwedge(E_3)$	139
2.7.2	The Hilbert Space $\bigwedge(E_3^d)$	140
2.7.3	Multivectors	142
2.8	The Clifford Structure of the Grassmann Algebra (Exterior–Interior Kähler Algebra)	144
2.8.1	The Kähler Algebra $\bigwedge(E_3)_\vee$	144
2.8.2	The Kähler Algebra $\bigwedge(E_3^d)_\vee$	145
2.9	The C^* -Algebra $\text{End}(E_3)$ of the Euclidean Space	145
2.10	Linear Operator Equations	146

2.10.1	The Prototype	146
2.10.2	The Grassmann Theorem	148
2.10.3	The Superposition Principle	151
2.10.4	Duality and the Fredholm Alternative	153
2.10.5	The Language of Matrices	157
2.10.6	The Gaussian Elimination Method	163
2.11	Changing the Basis and the Cobasis	164
2.11.1	Similarity of Matrices	165
2.11.2	Volume Functions	166
2.11.3	The Determinant of a Linear Operator	167
2.11.4	The Reciprocal Basis in Crystallography	168
2.11.5	Dual Pairing	170
2.11.6	The Trace of a Linear Operator	170
2.11.7	The Dirac Calculus	171
2.12	The Strategy of Quotient Algebras and Universal Properties	174
2.13	A Glance at Division Algebras	176
2.13.1	From Real Numbers to Cayley's Octonions	176
2.13.2	Uniqueness Theorems	177
2.13.3	The Fundamental Dimension Theorem	178
3.	Representations of Symmetries in Mathematics and	
	Physics	181
3.1	The Symmetric Group as a Prototype	181
3.2	Incredible Cancellations	184
3.3	The Symmetry Strategy in Mathematics and Physics	186
3.4	Lie Groups and Lie Algebras	187
3.5	Basic Notions of Representation Theory	189
3.5.1	Linear Representations of Groups	189
3.5.2	Linear Representations of Lie Algebras	193
3.6	The Reflection Group \mathcal{Z}_2 as a Prototype	194
3.6.1	Representations of \mathcal{Z}_2	194
3.6.2	Parity of Elementary Particles	195
3.6.3	Reflections and Chirality in Nature	196
3.6.4	Parity Violation in Weak Interaction	196
3.6.5	Helicity	196
3.7	Permutation of Elementary Particles	197
3.7.1	The Principle of Indistinguishability of Quantum Particles	197
3.7.2	The Pauli Exclusion Principle	197
3.7.3	Entangled Quantum States	198
3.8	The Diagonalization of Linear Operators	199
3.8.1	The Theorem of Principal Axes in Geometry and in Quantum Theory	200
3.8.2	The Schur Lemma in Linear Representation Theory	202
3.8.3	The Jordan Normal Form of Linear Operators	202

3.8.4	The Standard Maximal Torus of the Lie Group $SU(n)$ and the Standard Cartan Subalgebra of the Lie Algebra $su(n)$	204
3.8.5	Eigenvalues and the Operator Strategy for Lie Algebras (Adjoint Representation)	204
3.9	The Action of a Group on a Physical State Space, Orbits, and Gauge Theory	205
3.10	The Intrinsic Symmetry of a Group	206
3.11	Linear Representations of Finite Groups and the Hilbert Space of Functions on the Group	207
3.12	The Tensor Product of Representations and Characters ...	211
3.13	Applications to the Symmetric Group $Sym(n)$	214
3.13.1	The Characters of the Symmetric Group $Sym(2)$...	214
3.13.2	The Characters of the Symmetric Group $Sym(3)$...	216
3.13.3	Partitions and Young Frames	217
3.13.4	Young Tableaux and the Construction of a Complete System of Irreducible Representations	222
3.14	Application to the Standard Model in Elementary Particle Physics	225
3.14.1	Quarks and Baryons	225
3.14.2	Antiquarks and Mesons	236
3.14.3	The Method of Highest Weight for Composed Particles	239
3.14.4	The Pauli Exclusion Principle and the Color of Quarks	241
3.15	The Complexification of Lie Algebras	244
3.15.1	Basic Ideas	246
3.15.2	The Complex Lie Algebra $sl_{\mathbb{C}}(3, \mathbb{C})$ and Root Functionals	248
3.15.3	Representations of the Complex Lie Algebra $sl_{\mathbb{C}}(3, \mathbb{C})$ and Weight Functionals	252
3.16	Classification of Groups	253
3.16.1	Simplicity	253
3.16.2	Direct Product and Semisimplicity	255
3.16.3	Solvability	255
3.16.4	Semidirect Product	256
3.17	Classification of Lie Algebras	259
3.17.1	The Classification of Complex Simple Lie Algebras .	259
3.17.2	Semisimple Lie Algebras	261
3.17.3	Solvability and the Heisenberg Algebra in Quantum Mechanics	263
3.17.4	Semidirect Product and the Levi Decomposition ...	264
3.17.5	The Casimir Operators	266
3.18	Symmetric and Antisymmetric Functions	267

3.18.1	Symmetrization and Antisymmetrization	268
3.18.2	Elementary Symmetric Polynomials	270
3.18.3	Power Sums	271
3.18.4	Completely Symmetric Polynomials	271
3.18.5	Symmetric Schur Polynomials	272
3.18.6	Raising Operators and the Creation and Annihilation of Particles	274
3.19	Formal Power Series Expansions and Generating Functions ..	275
3.19.1	The Fundamental Frobenius Character Formula	276
3.19.2	The Pfaffian	278
3.20	Frobenius Algebras and Frobenius Manifolds	278
3.21	Historical Remarks	279
3.22	Supersymmetry	287
3.22.1	Graduation in Nature	287
3.22.2	General Strategy in Mathematics	287
3.22.3	The Super Lie Algebra of the Euclidean Space	288
3.23	Artin's Braid Group	290
3.23.1	The Braid Relation	290
3.23.2	The Yang-Baxter Equation	291
3.23.3	The Geometric Meaning of the Braid Group	292
3.23.4	The Topology of the State Space of n Indistinguish- able Particles in the Plane	294
3.24	The HOMFLY Polynomials in Knot Theory	295
3.25	Quantum Groups	297
3.25.1	Quantum Mechanics as a Deformation	297
3.25.2	Manin's Quantum Planes \mathbb{R}_q^2 and \mathbb{C}_q^2	298
3.25.3	The Coordinate Algebra of the Lie Group $SL(2, \mathbb{C})$..	300
3.25.4	The Quantum Group $SL_q(2, \mathbb{C})$	301
3.25.5	The Quantum Algebra $sl_q(2, \mathbb{C})$	302
3.25.6	The Coaction of the Quantum Group $SL_q(2, \mathbb{C})$ on the Quantum Plane \mathbb{C}_q^2	303
3.25.7	Noncommutative Euclidean Geometry and Quantum Symmetry	304
3.26	Additive Groups, Betti Numbers, Torsion Coefficients, and Homological Products	306
3.27	Lattices and Modules	309
4.	The Euclidean Manifold \mathbb{E}^3	321
4.1	Velocity Vectors and the Tangent Space	321
4.2	Duality and Cotangent Spaces	323
4.3	Parallel Transport and Acceleration	323
4.4	Newton's Law of Motion	324
4.5	Bundles Over the Euclidean Manifold	324
4.5.1	The Tangent Bundle and Velocity Vector Fields	325
4.5.2	The Cotangent Bundle and Covector Fields	325

4.5.3	Tensor Bundles and Tensor Fields	326
4.5.4	The Frame Bundle	327
4.6	Historical Remarks	327
4.6.1	Newton and Leibniz	327
4.6.2	The Lebesgue Integral	329
4.6.3	The Dirac Delta Function and Laurent Schwartz's Distributions	330
4.6.4	The Algebraization of the Calculus	330
4.6.5	Formal Power Series Expansions and the Ritt Theorem	331
4.6.6	Differential Rings and Derivations	331
4.6.7	The p -adic Numbers	332
4.6.8	The Local-Global Principle in Mathematics	336
4.6.9	The Global Adelic Ring	337
4.6.10	Solenoids, Foliations, and Chaotic Dynamical Systems	339
4.6.11	Period Three Implies Chaos	345
4.6.12	Differential Calculi, Noncommutative Geometry, and the Standard Model in Particle Physics	346
4.6.13	BRST-Symmetry, Cohomology, and the Quantiza- tion of Gauge Theories	347
4.6.14	Itô's Stochastic Calculus	348
5.	The Lie Group $U(1)$ as a Paradigm in Harmonic Analysis and Geometry	355
5.1	Linearization and the Lie Algebra $u(1)$	355
5.2	The Universal Covering Group of $U(1)$	356
5.3	Left-Invariant Velocity Vector Fields on $U(1)$	356
5.3.1	The Maurer-Cartan Form of $U(1)$	357
5.3.2	The Maurer-Cartan Structural Equation	358
5.4	The Riemannian Manifold $U(1)$ and the Haar Measure	358
5.5	The Discrete Fourier Transform	359
5.5.1	The Hilbert Space $L_2(U(1))$	359
5.5.2	Pseudo-Differential Operators	360
5.5.3	The Sobolev Space $W_2^m(U(1))$	361
5.6	The Group of Motions on the Gaussian Plane	361
5.7	Rotations of the Euclidean Plane	362
5.8	Pontryagin Duality for $U(1)$ and Quantum Groups	369
6.	Infinitesimal Rotations and Constraints in Physics	371
6.1	The Group $U(E_3)$ of Unitary Transformations	371
6.2	Euler's Rotation Formula	373
6.3	The Lie Algebra of Infinitesimal Rotations	374
6.4	Constraints in Classical Physics	375
6.4.1	Archimedes' Lever Principle	375

6.4.2	d'Alembert's Principle of Virtual Power	377
6.4.3	d'Alembert's Principle of Virtual Work	378
6.4.4	The Gaussian Principle of Least Constraint and Constraining Forces	378
6.4.5	Manifolds and Lagrange's Variational Principle	383
6.4.6	The Method of Perturbation Theory	384
6.4.7	Further Reading on Perturbation Theory and its Applications	385
6.5	Application to the Motion of a Rigid Body	388
6.5.1	The Center of Gravity	389
6.5.2	Moving Orthonormal Frames and Infinitesimal Rotations	389
6.5.3	Kinetic Energy and the Inertia Tensor	391
6.5.4	The Equations of Motion – the Existence and Uniqueness Theorem	393
6.5.5	Euler's Equation of the Spinning Top	395
6.5.6	Equilibrium States and Torque	397
6.5.7	The Principal Bundle $\mathbb{R}^3 \times SO(3)$ – the Position Space of a Rigid Body	397
6.6	A Glance at Constraints in Quantum Field Theory	398
6.6.1	Gauge Transformations and Virtual Degrees of Freedom in Gauge Theory	399
6.6.2	Elimination of Unphysical States (Ghosts).....	400
6.6.3	Degenerate Minimum Problems	401
6.6.4	Variation of the Action Functional	404
6.6.5	Degenerate Lagrangian and Constraints.....	408
6.6.6	Degenerate Legendre Transformation	408
6.6.7	Global and Local Symmetries.....	411
6.6.8	Quantum Symmetries and Anomalies	414
6.7	Perspectives	417
6.7.1	Topological Constraints in Maxwell's Theory of Electromagnetism	417
6.7.2	Constraints in Einstein's Theory of General Relativity	417
6.7.3	Hilbert's Algebraic Theory of Relations (Syzygies) .	417
6.8	Further Reading	418
7.	Rotations, Quaternions, the Universal Covering Group, and the Electron Spin	425
7.1	Quaternions and the Cayley–Hamilton Rotation Formula ..	425
7.2	The Universal Covering Group $SU(2)$	426
7.3	Irreducible Unitary Representations of the Group $SU(2)$ and the Spin	427
7.3.1	The Spin Quantum Numbers	428
7.3.2	The Addition Theorem for the Spin	434

7.3.3	The Model of Homogeneous Polynomials	435
7.3.4	The Clebsch–Gordan Coefficients	436
7.4	Heisenberg’s Isospin	437
8.	Changing Observers – A Glance at Invariant Theory	
	Based on the Principle of the Correct Index Picture	439
8.1	A Glance at the History of Invariant Theory	439
8.2	The Basic Philosophy	440
8.3	The Mnemonic Principle of the Correct Index Picture	443
8.4	Real-Valued Physical Fields	444
8.4.1	The Chain Rule and the Key Duality Relation	445
8.4.2	Linear Differential Operators	446
8.4.3	Duality and Differentials	447
8.4.4	Admissible Systems of Observers	449
8.4.5	Tensorial Families and the Construction of Invariants via the Basic Trick of Index Killing	452
8.4.6	Orientation, Pseudo-Tensorial Families, and the Levi-Civita Duality	460
8.5	Differential Forms (Exterior Product)	464
8.5.1	Cartan Families and the Cartan Differential	464
8.5.2	Hodge Duality, the Hodge Codifferential, and the Laplacian (Hodge’s Star Operator)	469
8.6	The Kähler–Clifford Calculus and the Dirac Operator (Interior Product)	473
8.6.1	The Exterior Differential Algebra	475
8.6.2	The Interior Differential Algebra	477
8.6.3	Kähler Duality	479
8.6.4	Applications to Fundamental Differential Equations in Physics	480
8.6.5	The Potential Equation and the Importance of the de Rham Cohomology	481
8.6.6	Tensorial Differential Forms	482
8.7	Integrals over Differential Forms	483
8.8	Derivatives of Tensorial Families	484
8.8.1	The Lie Algebra of Linear Differential Operators and the Lie Derivative	487
8.8.2	The Inverse Index Principle	493
8.8.3	The Covariant Derivative (Weyl’s Affine Con- nection)	494
8.9	The Riemann–Weyl Curvature Tensor	503
8.9.1	Second-Order Covariant Partial Derivatives	504
8.9.2	Local Flatness	506
8.9.3	The Method of Differential Forms (Cartan’s Struc- tural Equations)	507
8.9.4	The Operator Method	510

8.10	The Riemann–Christoffel Curvature Tensor	511
8.10.1	The Levi-Civita Metric Connection	512
8.10.2	Levi-Civita’s Parallel Transport	513
8.10.3	Symmetry Properties of the Riemann–Christoffel Curvature Tensor	515
8.10.4	The Ricci Curvature Tensor and the Einstein Tensor	516
8.10.5	The Conformal Weyl Curvature Tensor	517
8.10.6	The Hodge Codifferential and the Covariant Partial Derivative	519
8.10.7	The Weitzenböck Formula for the Hodge Laplacian .	519
8.10.8	The One-Dimensional σ -Model and Affine Geodesics	520
8.11	The Beauty of Connection-Free Derivatives	522
8.11.1	The Lie Derivative	523
8.11.2	The Cartan Derivative	523
8.11.3	The Weyl Derivative	524
8.12	Global Analysis	526
8.13	Summary of Notation	527
8.14	Two Strategies in Invariant Theory	529
8.15	Intrinsic Tangent Vectors and Derivations	529
8.16	Further Reading on Symmetry and Invariants	534
9.	Applications of Invariant Theory to the Rotation Group .	557
9.1	The Method of Orthonormal Frames on the Euclidean Manifold	557
9.1.1	Hamilton’s Quaternionic Analysis	557
9.1.2	Transformation of Orthonormal Frames	559
9.1.3	The Coordinate-Dependent Approach ($SO(3)$ -Tensor Calculus)	560
9.1.4	The Coordinate-Free Approach	561
9.1.5	Hamilton’s Nabla Calculus	563
9.1.6	Rotations and Cauchy’s Invariant Functions	565
9.2	Curvilinear Coordinates	567
9.2.1	Local Observers	567
9.2.2	The Metric Tensor	568
9.2.3	The Volume Form	569
9.2.4	Special Coordinates	569
9.3	The Index Principle of Mathematical Physics	574
9.3.1	The Basic Trick	574
9.3.2	Applications to Vector Analysis	575
9.4	The Euclidean Connection and Gauge Theory	576
9.4.1	Covariant Partial Derivative	577
9.4.2	Curves of Least Kinectic Energy (Affine Geodesics) .	577
9.4.3	Curves of Minimal Length	579
9.4.4	The Gauss Equations of Moving Frames	580

9.4.5	Parallel Transport of a Velocity Vector and Cartan's Propagator Equation	581
9.4.6	The Dual Cartan Equations of Moving Frames	584
9.4.7	Global Parallel Transport on Lie Groups and the Maurer–Cartan Form	585
9.4.8	Cartan's Global Connection Form on the Frame Bundle of the Euclidean Manifold	587
9.4.9	The Relation to Gauge Theory	590
9.4.10	The Reduction of the Frame Bundle to the Orthonormal Frame Bundle	593
9.5	The Sphere as a Paradigm in Riemannian Geometry and Gauge Theory	593
9.5.1	The Newtonian Equation of Motion and Levi-Civita's Parallel Transport	595
9.5.2	Geodesic Triangles and the Gaussian Curvature	599
9.5.3	Geodesic Circles and the Gaussian Curvature	600
9.5.4	The Spherical Pendulum	600
9.5.5	Geodesics and Gauge Transformations	603
9.5.6	The Local Hilbert Space Structure	606
9.5.7	The Almost Complex Structure	607
9.5.8	The Levi-Civita Connection on the Tangent Bundle and the Riemann Curvature Tensor	608
9.5.9	The Components of the Riemann Curvature Tensor and Gauge Fixing	617
9.5.10	Computing the Riemann Curvature Operator via Parallel Transport Along Loops	619
9.5.11	The Connection on the Frame Bundle and Parallel Transport	620
9.5.12	Poincaré's Topological No-Go Theorem for Velocity Vector Fields on a Sphere	623
9.6	Gauss' Theorema Egregium	623
9.6.1	The Natural Basis and Cobasis	623
9.6.2	Intrinsic Metric Properties	627
9.6.3	The Extrinsic Definition of the Gaussian Curvature	628
9.6.4	The Gauss–Weingarten Equations for Moving Frames	630
9.6.5	The Integrability Conditions and the Riemann Curvature Tensor	631
9.6.6	The Intrinsic Characterization of the Gaussian Curvature (Theorema Egregium)	632
9.6.7	Differential Invariants and the Existence and Uniqueness Theorem of Classical Surface Theory ...	633
9.6.8	Gauss' Theorema Elegantissimum and the Gauss–Bonnet Theorem	634

9.6.9	Gauss' Total Curvature and Topological Charges . . .	635
9.6.10	Cartan's Method of Moving Orthonormal Frames . .	636
9.7	Parallel Transport in Physics	638
9.8	Finsler Geometry	638
9.9	Further Reading	640
10.	Temperature Fields on the Euclidean Manifold \mathbb{E}^3	645
10.1	The Directional Derivative	645
10.2	The Lie Derivative of a Temperature Field along the Flow of Fluid Particles	647
10.2.1	The Flow	647
10.2.2	The Linearized Flow	650
10.2.3	The Lie Derivative	651
10.2.4	Conservation Laws	652
10.3	Higher Variations of a Temperature Field and the Taylor Expansion	652
10.4	The Fréchet Derivative	653
10.5	Global Linearization of Smooth Maps and the Tangent Bundle	654
10.6	The Global Chain Rule	657
10.7	The Transformation of Temperature Fields	657
11.	Velocity Vector Fields on the Euclidean Manifold \mathbb{E}^3	659
11.1	The Transformation of Velocity Vector Fields	661
11.2	The Lie Derivative of an Electric Field along the Flow of Fluid Particles	663
11.2.1	The Lie Derivative	663
11.2.2	Conservation Laws	663
11.2.3	The Lie Algebra of Velocity Vector Fields	664
12.	Covector Fields and Cartan's Exterior Differential – the Beauty of Differential Forms	665
12.1	Ariadne's Thread	666
12.1.1	One Dimension	666
12.1.2	Two Dimensions	670
12.1.3	Three Dimensions	677
12.1.4	Integration over Manifolds	681
12.1.5	Integration over Singular Chains	684
12.2	Applications to Physics	685
12.2.1	Single-Valued Potentials and Gauge Transformations	685
12.2.2	Multi-Valued Potentials and Riemann Surfaces	687
12.2.3	The Electrostatic Coulomb Force and the Dirac Delta Distribution	690
12.2.4	The Magic Green's Function and the Dirac Delta Distribution	691

12.2.5	Conservation of Heat Energy – the Paradigm of Conservation Laws in Physics	695
12.2.6	The Classical Predecessors of the Yang–Mills Equations in Gauge Theory (Fluid Dynamics and Electrodynamics)	698
12.2.7	Thermodynamics and the Pfaff Problem	698
12.2.8	Classical Mechanics and Symplectic Geometry	700
12.2.9	The Universality of Differential Forms	700
12.2.10	Cartan’s Covariant Differential and the Four Fundamental Interactions in Nature	700
12.3	Cartan’s Algebra of Alternating Differential Forms	701
12.3.1	The Geometric Approach	701
12.3.2	The Grassmann Bundle	704
12.3.3	The Tensor Bundle	705
12.3.4	The Transformation of Covector Fields	705
12.4	Cartan’s Exterior Differential	706
12.4.1	Invariant Definition via the Lie Algebra of Velocity Vector Fields	707
12.4.2	The Supersymmetric Leibniz Rule	709
12.4.3	The Poincaré Cohomology Rule	710
12.4.4	The Axiomatic Approach	710
12.5	The Lie Derivative of Differential Forms	712
12.5.1	Invariant Definition via the Flow of Fluid Particles ..	712
12.5.2	The Contraction Product between Velocity Vector Fields and Differential Forms	714
12.5.3	Cartan’s Magic Formula	714
12.5.4	The Lie Derivative of the Volume Form	715
12.5.5	The Lie Derivative of the Metric Tensor Field	715
12.5.6	The Lie Derivative of Linear Operator Fields	716
12.6	Diffeomorphisms and the Mechanics of Continua – the Prototype of an Effective Theory in Physics	717
12.6.1	Linear Diffeomorphisms and Deformation Operators	718
12.6.2	Local Diffeomorphisms	719
12.6.3	Proper Maps and Hadamard’s Theorem on Diffeomorphisms	720
12.6.4	Monotone Operators and Diffeomorphisms	720
12.6.5	Sard’s Theorem on the Genericity of Regular Solution Sets	721
12.6.6	The Strain Tensor and the Stress Tensor in Cauchy’s Theory of Elasticity	722
12.6.7	The Rate-of-Strain Tensor and the Stress Tensor in the Hydrodynamics of Viscous Fluids	725
12.6.8	Vorticity Lines of a Fluid	728
12.6.9	The Lie Derivative of the Covector Field	728

12.7	The Generalized Stokes Theorem (Main Theorem of Calculus)	729
12.8	Conservation Laws	731
12.8.1	Infinitesimal Isometries (Metric Killing Vector Fields)	732
12.8.2	Absolute Integral Invariants and Incompressible Fluids	734
12.8.3	Relative Integral Invariants and the Vorticity Theorems for Fluids due to Thomson and Helmholtz	735
12.8.4	The Transport Theorem	735
12.8.5	The Noether Theorem – Symmetry Implies Conservation Laws in the Calculus of Variations ...	737
12.9	The Hamiltonian Flow on the Euclidean Manifold – a Paradigm of Hamiltonian Mechanics	744
12.9.1	Hamilton’s Principle of Critical Action	746
12.9.2	Basic Formulas	748
12.9.3	The Poincaré–Cartan Integral Invariant	749
12.9.4	Energy Conservation and the Liouville Integral Invariant	749
12.9.5	Jacobi’s Canonical Transformations, Lie’s Contact Geometry, and Symplectic Geometry	750
12.9.6	Hilbert’s Invariant Integral	753
12.9.7	Jacobi’s Integration Method	753
12.9.8	Legendre Transformation	754
12.9.9	Carathéodory’s Royal Road to the Calculus of Variations	755
12.9.10	Geometrical Optics	759
12.10	The Main Theorems in Classical Gauge Theory (Existence of Potentials)	760
12.10.1	Contractible Manifolds (the Poincaré–Volterra Theorem)	762
12.10.2	Non-Contractible Manifolds and Betti Numbers (De Rham’s Theorem on Periods)	764
12.10.3	The Main Theorem for Velocity Vector Fields	766
12.11	Systems of Differential Forms	767
12.11.1	Integrability Condition	767
12.11.2	The Frobenius Theorem for Pfaff Systems	769
12.11.3	The Dual Frobenius Theorem	770
12.11.4	The Pfaff Normal Form and the Second Law of Thermodynamics	770
12.12	Hodge Duality	771
12.12.1	The Hodge Codifferential	772
12.12.2	The Hodge Homology Rule	773

12.12.3 The Relation between the Cartan–Hodge Calculus and Classical Vector Analysis via Riesz Duality	773
12.12.4 The Classical Prototype of the Yang–Mills Equation in Gauge Theory	774
12.12.5 The Hodge–Laplace Operator and Harmonic Forms.	775
12.13 Further Reading	775
12.14 Historical Remarks	777

Part II. Ariadne’s Thread in Gauge Theory

13. The Commutative Weyl $U(1)$-Gauge Theory and the Electromagnetic Field	811
13.1 Basic Ideas	811
13.2 The Fundamental Principle of Local Symmetry Invariance in Modern Physics	814
13.2.1 The Free Meson	814
13.2.2 Local Symmetry and the Charged Meson in an Electromagnetic Field	818
13.3 The Vector Bundle $\mathbb{M}^4 \times \mathbb{C}$, Covariant Directional Derivative, and Curvature	820
13.4 The Principal Bundle $\mathbb{M}^4 \times U(1)$ and the Parallel Transport of the Local Phase Factor	825
13.5 Parallel Transport of Physical Fields – the Propagator Approach	827
13.6 The Wilson Loop and Holonomy	829
14. Symmetry Breaking	831
14.1 The Prototype in Mechanics	831
14.2 The Goldstone-Particle Mechanism	832
14.3 The Higgs-Particle Mechanism	834
14.4 Dimensional Reduction and the Kaluza–Klein Approach ...	835
14.5 Superconductivity and the Ginzburg–Landau Equation	836
14.6 The Idea of Effective Theories in Physics	840
15. The Noncommutative Yang–Mills $SU(N)$-Gauge Theory	843
15.1 The Vector Bundle $\mathbb{M}^4 \times \mathbb{C}^N$, Covariant Directional Derivative, and Curvature	843
15.2 The Principal Bundle $\mathbb{M}^4 \times \mathcal{G}$ and the Parallel Transport of the Local Phase Factor	847
15.3 Parallel Transport of Physical Fields – the Propagator Approach	852
15.4 The Principle of Critical Action and the Yang–Mills Equations	854
15.5 The Universal Extension Strategy via the Leibniz Rule	858

15.6	Tensor Calculus on Vector Bundles	859
15.6.1	Tensor Algebra	860
15.6.2	Connection and Christoffel Symbols	863
15.6.3	Covariant Differential for Differential Forms of Tensor Type	864
15.6.4	Application to the Riemann Curvature Operator ...	867
16.	Cocycles and Observers	871
16.1	Cocycles	871
16.2	Physical Fields via the Cocycle Strategy	872
16.3	Local Phase Factors via the Cocycle Strategy	873
17.	The Axiomatic Geometric Approach to Bundles	875
17.1	Connection on a Vector Bundle	875
17.2	Connection on a Principal Bundle	879
17.3	The Philosophy of Parallel Transport	883
17.3.1	Vector Bundles Associated to a Principal Bundle ...	884
17.3.2	Horizontal Vector Fields on a Principal Bundle	887
17.3.3	The Lifting of Curves in Fiber Bundles	888
17.4	A Glance at the History of Gauge Theory	891
17.4.1	From Weyl's Gauge Theory in Gravity to the Standard Model in Particle Physics	891
17.4.2	From Gauss' Theorema Egregium to Modern Differential Geometry	896
17.4.3	The Work of Hermann Weyl	900

Part III. Einstein's Theory of Special Relativity

18.	Inertial Systems and Einstein's Principle of Special Relativity	905
18.1	The Principle of Special Relativity	908
18.1.1	The Lorentz Boost	909
18.1.2	The Transformation of Velocities	910
18.1.3	Time Dilatation	911
18.1.4	Length Contraction	911
18.1.5	The Synchronization of Clocks	912
18.1.6	General Change of Inertial Systems in Terms of Physics	912
18.2	Matrix Groups	914
18.2.1	The Group $O(1, 1)$	914
18.2.2	The Lorentz Group $O(1, 3)$	916
18.3	Infinitesimal Transformations	918
18.3.1	The Lie Algebra $\mathfrak{o}(1, 3)$ of the Lorentz Group $O(1, 3)$	918
18.3.2	The Lie Algebra $\mathfrak{p}(1, 3)$ of the Poincaré Group $P(1, 3)$	921

18.4	The Minkowski Space M_4	923
18.4.1	Pseudo-Orthonormal Systems and Inertial Systems .	923
18.4.2	Orientation	926
18.4.3	Proper Time and the Twin Paradox	926
18.4.4	The Free Relativistic Particle and the Energy-Mass Equivalence	927
18.4.5	The Photon	929
18.5	The Minkowski Manifold M^4	929
18.5.1	Causality and the Maximal Signal Velocity	930
18.5.2	Hodge Duality	931
18.5.3	Arbitrary Local Coordinates	932
19.	The Relativistic Invariance of the Maxwell Equations	935
19.1	Historical Background	936
19.1.1	The Coulomb Force and the Gauss Law	937
19.1.2	The Ampère Force and the Ampère Law	941
19.1.3	Joule's Heat Energy Law	944
19.1.4	Faraday's Induction Law	944
19.1.5	Electric Dipoles	945
19.1.6	Magnetic Dipoles	947
19.1.7	The Electron Spin	948
19.1.8	The Dirac Magnetic Monopole	951
19.1.9	Vacuum Polarization in Quantum Electrodynamics .	952
19.2	The Maxwell Equations in a Vacuum	954
19.2.1	The Global Maxwell Equations Based on Electric and Magnetic Flux	955
19.2.2	The Local Maxwell Equations Formulated in Maxwell's Language of Vector Calculus	957
19.2.3	Discrete Symmetries and <i>CPT</i>	958
19.3	Invariant Formulation of the Maxwell Equations in a Vacuum	960
19.3.1	Einstein's Language of Tensor Calculus	960
19.3.2	The Language of Differential Forms and Hodge Duality	962
19.3.3	De Rham Cohomology and the Four-Potential of the Electromagnetic Field	964
19.3.4	The Language of Fiber Bundles	967
19.4	The Transformation Law for the Electromagnetic Field	967
19.5	Electromagnetic Waves	969
19.6	Invariants of the Electromagnetic Field	969
19.6.1	The Motion of a Charged Particle and the Lorentz Force	970
19.6.2	The Energy Density and the Energy-Momentum Tensor	971
19.6.3	Conservation Laws	972

19.7	The Principle of Critical Action	976
19.7.1	The Electromagnetic Field	976
19.7.2	Motion of Charged Particles and Gauge Transformations	977
19.8	Weyl Duality and the Maxwell Equations in Materials	979
19.8.1	The Maxwell Equations in the Rest System	980
19.8.2	Typical Examples of Constitutive Laws	980
19.8.3	The Maxwell Equations in an Arbitrary Inertial System	982
19.9	Physical Units	983
19.9.1	The SI System	983
19.9.2	The Universal Approach	985
19.10	Further Reading	986
20.	The Relativistic Invariance of the Dirac Equation and the Electron Spin	995
20.1	The Dirac Equation	995
20.2	Changing the Inertial System	997
20.3	The Electron Spin	999
<hr/>		
Part IV. Ariadne's Thread in Cohomology		
<hr/>		
21.	The Language of Exact Sequences	1003
21.1	Applications to Linear Algebra	1003
21.2	The Fredholm Alternative	1005
21.3	The Deviation from Exact Sequences and Cohomology	1007
21.4	Perspectives	1007
22.	Electrical Circuits as a Paradigm in Homology and Cohomology	1009
22.1	Basic Equations	1009
22.2	Euler's Bridge Problem and the Kirchhoff Rules	1015
22.3	Weyl's Theorem on Electrical Circuits	1018
22.4	Homology and Cohomology in Electrical Circuits	1019
22.5	Euler Characteristic and Betti Numbers	1024
22.6	The Discrete de Rham Theory	1026
23.	The Electromagnetic Field and the de Rham Cohomology	1027
23.1	The De Rham Cohomology Groups	1027
23.1.1	Elementary Examples	1027
23.1.2	Advanced Examples	1030
23.1.3	Topological Invariance of the de Rham Cohomology Groups	1032

23.1.4	Homotopical Invariance of the de Rham Cohomology Groups	1033
23.2	The Fundamental Potential Equation in Gauge Theory and the Analytic Meaning of the Betti Numbers	1036
23.3	Hodge Theory (Representing Cohomology Classes by Harmonic Forms)	1038
23.4	The Topology of the Electromagnetic Field and Potentials .	1039
23.5	The Analysis of the Electromagnetic Field	1045
23.5.1	The Main Theorem of Electrostatics, the Dirichlet Principle, and Generalized Functions	1045
23.5.2	The Coulomb Gauge and the Main Theorem of Magnetostatics	1048
23.5.3	The Main Theorem of Electrodynamics	1052
23.6	Important Tools	1053
23.6.1	The Exact Mayer–Vietoris Sequence and the Computation of the de Rham Cohomology Groups	1053
23.6.2	The de Rham Cohomology Algebra.....	1055
23.7	The Beauty of Partial Differential Equations in Physics, Analysis, and Topology	1055
23.8	A Glance at Topological Quantum Field Theory (Statistics for Mathematical Structures)	1056
23.9	Further Reading	1061
Appendix		1069
A.1	Manifolds and Diffeomorphisms	1069
A.1.1	Manifolds without Boundary	1071
A.1.2	Manifolds with Boundary	1072
A.1.3	Submanifolds	1076
A.1.4	Partition of Unity and the Globalization of Physical Fields	1077
A.2	The Solution of Nonlinear Equations	1078
A.2.1	Linearization and the Rank Theorem	1078
A.2.2	Violation of the Rank Condition and Bifurcation ...	1082
A.3	Lie Matrix Groups	1083
A.4	The Main Theorem on the Global Structure of Lie Groups .	1085
Epilogue		1087
References		1089
List of Symbols		1091
Index		1099