Contents

Preface vii

Acknowledgments ix

Using This Book xi

PART I: EXAMPLES OF VARIETIES AND MAPS

LECTURE 1
Affine and Projective Varieties 3

A Note About Our Field 3
Affine Space and Affine Varieties 3
Projective Space and Projective Varieties 3
Linear Spaces 5
Finite Sets 6
Hypersurfaces 8
Analytic Subvarieties and Submanifolds 8
The Twisted Cubic 9
Rational Normal Curves 10
Determinantal Representation of the Rational Normal Curve 11
Another Parametrization of the Rational Normal Curve 11
The Family of Plane Conics 12
A Synthetic Construction of the Rational Normal Curve 13
Other Rational Curves 14
Varieties Defined over Subfields of K 16
A Note on Dimension, Smoothness, and Degree 16

LECTURE 2
Regular Functions and Maps 17

The Zariski Topology 17
Regular Functions on an Affine Variety 18

Projective Varieties 20
Regular Maps 21
The Veronese Map 23
Determinantal Representation of Veronese Varieties 24
Subvarieties of Veronese Varieties 24
The Segre Maps 25
Subvarieties of Segre Varieties 27
Products of Varieties 28
Graphs 29
Fiber Products 30
Combinations of Veronese and Segre Maps 30

LECTURE 3
Cones, Projections, and More About Products 32

Cones 32
Quadrics 33
Projections 34
More Cones 37
More Projections 38
Constructible Sets 39

LECTURE 4
Families and Parameter Spaces 41

Families of Varieties 41
The Universal Hyperplane 42
The Universal Hyperplane Section 43
Parameter Spaces of Hypersurfaces 44
Universal Families of Hypersurfaces 45
A Family of Lines 47

LECTURE 5
Ideals of Varieties, Irreducible Decomposition, and the Nullstellensatz 48

Generating Ideals 48
Ideals of Projective Varieties 50
Irreducible Varieties and Irreducible Decomposition 51
General Objects 53
General Projections 54
General Twisted Cubics 55
Double Point Loci 56
A Little Algebra 57
Restatements and Corollaries 60

LECTURE 6
Grassmannians and Related Varieties 63

Grassmannians 63
Subvarieties of Grassmannians 66

The Grassmannian $\mathbb{G}(1, 3)$ 67
An Analog of the Veronese Map 68
Incidence Correspondences 68
Varieties of Incident Planes 69
The Join of Two Varieties 70
Fano Varieties 70

LECTURE 7
Rational Functions and Rational Maps 72

Rational Functions 72
Rational Maps 73
Graphs of Rational Maps 75
Birational Isomorphism 77
The Quadric Surface 78
Hypersurfaces 79
Degree of a Rational Map 79
Blow-Ups 80
Blowing Up Points 81
Blowing Up Subvarieties 82
The Quadric Surface Again 84
The Cubic Scroll in $\mathbb{P}^4$ 85
Unirationality 87

LECTURE 8
More Examples 88

The Join of Two Varieties 88
The Secant Plane Map 89
Secant Varieties 90
Trisecant Lines, etc. 90
Joins of Corresponding Points 91
Rational Normal Scrolls 92
Higher-Dimensional Scrolls 93
More Incidence Correspondences 94
Flag Manifolds 95
More Joins and Intersections 95
Quadrics of Rank 4 96
Rational Normal Scrolls II 97

LECTURE 9
Determinantal Varieties 98

Generic Determinantal Varieties 98
Segre Varieties 98
Secant Varieties of Segre Varieties 99
Linear Determinantal Varieties in General 99
Rational Normal Curves 100
Secant Varieties to Rational Normal Curves 103
Rational Normal Scrolls III 105

Rational Normal Scrolls IV 109
More General Determinantal Varieties 111
Symmetric and Skew-Symmetric Determinantal Varieties 112
Fano Varieties of Determinantal Varieties 112

LECTURE 10
Algebraic Groups 114

The General Linear Group GL_nK 114
The Orthogonal Group SO_nK 115
The Symplectic Group $Sp_{2n}K$ 116
Group Actions 116
$PGL_{n+1}K$ acts on $\mathbb{P}^n$ 116
PGL_2K Acts on $\mathbb{P}^2$ 117
PGL_2K Acts on $\mathbb{P}^3$ 118
PGL_2K Acts on $\mathbb{P}^n$ 119
PGL_3K Acts on $\mathbb{P}^5$ 120
PGL_3K Acts on $\mathbb{P}^9$ 121
PO_nK Acts on $\mathbb{P}^{n-1}$ (automorphisms of the Grassmannian) 122
PGL_nK Acts on $\mathbb{P}(\wedge^k K^n)$ 122
Quotients 123
Quotients of Affine Varieties by Finite Groups 124
Quotients of Affine Space 125
Symmetric Products 126
Quotients of Projective Varieties by Finite Groups 126
Weighted Projective Spaces 127

PART II: ATTRIBUTES OF VARIETIES

LECTURE 11
Definitions of Dimension and Elementary Examples 133

Hypersurfaces 136
Complete Intersections 136
Immediate Examples 138
The Universal k-Plane 142
Varieties of Incident Planes 142
Secant Varieties 143
Secant Varieties in General 146
Joins of Varieties 148
Flag Manifolds 148
(Some) Schubert Varieties 149

LECTURE 12
More Dimension Computations 151

Determinantal Varieties 151
Fano Varieties 152
Parameter Spaces of Twisted Cubics 155
Twisted Cubics 155

Twisted Cubics on a General Surface 156
Complete Intersections 157
Curves of Type (a, b) on a Quadric 158
Determinantal Varieties 159
Group Actions 161
$GL(V)$ Acts on $\mathrm{Sym}^d V$ and $\wedge^k V$ 161
$PGL_{n+1}K$ Acts on $(\mathbb{P}^n)^l$ and $\mathbb{G}(k, n)^l$ 161

LECTURE 13
Hilbert Polynomials 163

Hilbert Functions and Polynomials 163
Hilbert Function of the Rational Normal Curve 166
Hilbert Function of the Veronese Variety 166
Hilbert Polynomials of Curves 166
Syzygies 168
Three Points in $\mathbb{P}^2$ 170
Four Points in $\mathbb{P}^2$ 171
Complete Intersections: Koszul Complexes 172

LECTURE 14
Smoothness and Tangent Spaces 174

The Zariski Tangent Space to a Variety 174
A Local Criterion for Isomorphism 177
Projective Tangent Spaces 181
Determinantal Varieties 184

LECTURE 15
Gauss Maps, Tangential and Dual Varieties 186

A Note About Characteristic 186
Gauss Maps 188
Tangential Varieties 189
The Variety of Tangent Lines 190
Joins of Intersecting Varieties 193
The Locus of Bitangent Lines 195
Dual Varieties 196

LECTURE 16
Tangent Spaces to Grassmannians 200

Tangent Spaces to Grassmannians 200
Tangent Spaces to Incidence Correspondences 202
Varieties of Incident Planes 203
The Variety of Secant Lines 204
Varieties Swept out by Linear Spaces 204
The Resolution of the Generic Determinantal Variety 206
Tangent Spaces to Dual Varieties 208
Tangent Spaces to Fano Varieties 209

LECTURE 17
Further Topics Involving Smoothness and Tangent Spaces 211

Gauss Maps on Curves 211
Osculating Planes and Associated Maps 213
The Second Fundamental Form 214
The Locus of Tangent Lines to a Variety 215
Bertini's Theorem 216
Blow-ups, Nash Blow-ups, and the Resolution of Singularities 219
Subadditivity of Codimensions of Intersections 222

LECTURE 18
Degree 224

Bézout's Theorem 227
The Rational Normal Curves 229
More Examples of Degrees 231
Veronese Varieties 231
Segre Varieties 233
Degrees of Cones and Projections 234
Joins of Varieties 235
Unirationality of Cubic Hypersurfaces 237

LECTURE 19
Further Examples and Applications of Degree 239

Multidegree of a Subvariety of a Product 239
Projective Degree of a Map 240
Joins of Corresponding Points 241
Varieties of Minimal Degree 242
Degrees of Determinantal Varieties 243
Degrees of Varieties Swept out by Linear Spaces 244
Degrees of Some Grassmannians 245
Harnack's Theorem 247

LECTURE 20
Singular Points and Tangent Cones 251

Tangent Cones 251
Tangent Cones to Determinantal Varieties 256
Multiplicity 258
Examples of Singularities 260
Resolution of Singularities for Curves 264

LECTURE 21
Parameter Spaces and Moduli Spaces 266

Parameter Spaces 266
Chow Varieties 268
Hilbert Varieties 273

Curves of Degree 2 275
Moduli Spaces 278
Plane Cubics 279

LECTURE 22
Quadrics 282

Generalities about Quadrics 282
Tangent Spaces to Quadrics 283
Plane Conics 284
Quadric Surfaces 285
Quadrics in $\mathbb{P}^n$ 287
Linear Spaces on Quadrics 289
Lines on Quadrics 290
Planes on Four-Dimensional Quadrics 291
Fano Varieties of Quadrics in General 293
Families of Quadrics 295
The Variety of Quadrics in $\mathbb{P}^1$ 295
The Variety of Quadrics in $\mathbb{P}^2$ 296
Complete Conics 297
Quadrics in $\mathbb{P}^n$ 299
Pencils of Quadrics 301

Hints for Selected Exercises 308

References 314

Index 317