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Introduction to Charged Particle Beams

> "If you want to find the secrets of the universe, think in terms of energy, frequency, and vibration."
>
> —*Nikola Tesla*

In accelerator physics, a particle beam is typically defined as a collection of like-charged particles, all moving with momentum predominantly in one direction compared to the other two transverse directions. This characteristic allows the beam to be transported over long distances using electromagnetic fields and further accelerated to energies reaching several teraelectronvolts (TeV) in modern accelerators. Naturally occurring particle beams exist in space, commonly referred to as cosmic rays or solar particles. A specific example is the stream of charged particles, such as solar wind or proton beams, emitted by the Sun. These particles are captured by Earth's magnetic field, resulting in collisions with particles in Earth's upper atmosphere. In the Earth ionosphere, charged particles from the solar wind, guided by Earth's magnetic field, collide with oxygen and nitrogen atoms in the atmosphere, exciting them and releasing energy as colorful light displays known as the aurora, or Northern Lights.

Curiosity: *Have you ever wondered how particle accelerators can speed up tiny particles like protons or electrons to nearly the speed of light? Why do they need a vacuum—can't particles just move through air like anything else? To push particles forward, accelerators use special devices called RF cavities that work like swings, giving the particle a timed "kick" each cycle. But as particles gain speed, why do we need magnets to bend and focus their paths—why don't they just go straight? And did you know that your microwave oven uses a tiny kind of particle accelerator called a magnetron? What if, instead of electric fields, we could use sound waves or even gravity to accelerate particles—could that work someday? If batteries produce voltage, why can't we just use a giant one to reach GeV energy levels? Inside circular accelerators, how do particles manage to stay in sync with the accelerating fields without flying off-track? When too many particles gather close together, do they repel each other and cause the beam to spread out? It's amazing that the first particle accelerator, built in the 1930s, could fit on a tabletop! And why do fast-moving particles give off brilliant light—called synchrotron radiation—when they are forced to turn? Finally,*

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imagine this: could we one day shrink a whole accelerator down to fit on a tiny microchip? We try to explore some of these curious questions here—but many remain open, inviting you to keep wondering, exploring, and discovering.

The energy (E) of a particle [1] can be related to its temperature using the Boltzmann relation:

$$E = k_B T \tag{1.1}$$

$$k_B \approx 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$$

and 1 eV corresponds to approximately 11 605 K. This implies that particles accelerated to energies in the kiloelectron volt (keV), megaelectron volt (MeV), and gigaelectronvolt (GeV) range are effectively heated to extremely high temperatures. Understanding **particle beam dynamics** is crucial in space physics and accelerator sciences.

When ions are generated in a laboratory as a stream of charged particles, they must be accelerated and transported to a target with minimal intensity loss. These ions are used in various fields of science, such as nuclear physics, materials science, and atomic physics. In discussing heavy ions, which are larger than protons, a key difference lies in their approach to the speed of light as they gain energy. The total energy of a charged particle is given by the following equation:

The relativistic energy–momentum relation is:

$$E^2 = (pc)^2 + (m_0c^2)^2 \tag{1.2}$$

where E is the total energy of the particle, p is the relativistic momentum, m_0 is the rest mass of the particle, and c is the speed of light.

The relativistic momentum p is related to the velocity v of the particle by:

$$p = \gamma m_0 v \tag{1.3}$$

where γ (the Lorentz factor) is defined as:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{1.4}$$

Here $\beta = \frac{v}{c}$. Thus, the total energy of particle is given as follows:

$$E^2 = (\gamma m_0 vc)^2 + (m_0c^2)^2 = m_0^2c^4\left(\gamma^2\beta^2 + 1\right) = \gamma^2 m_0^2 c^4$$

$$E = \gamma m_0 c^2 \tag{1.5}$$

Finally, the kinetic energy K is given by:

$$K = E - m_0c^2 = \left(\frac{1}{\sqrt{1-\beta^2}} - 1\right) m_0c^2 \tag{1.6}$$

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \left(\frac{1}{1 + \frac{K}{m_0c^2}}\right)^2}$$

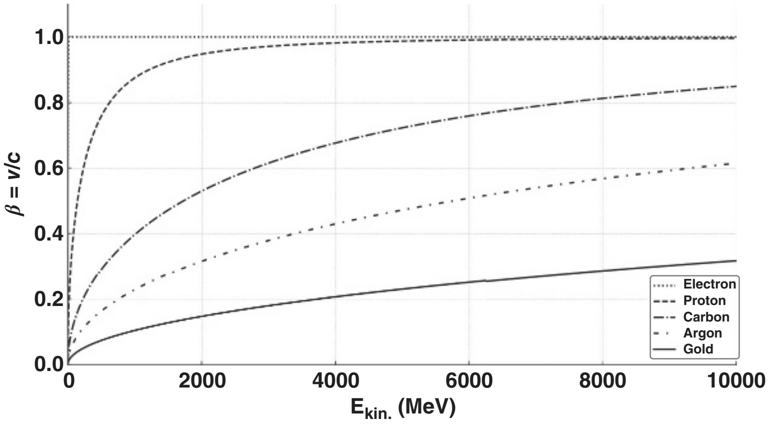

Figure 1.1 Velocity behavior of different charged particles as they approach the speed of light.

If we plot the velocity of different charged particles as per Equation 1.6, then the electrons begin to exhibit significant relativistic behavior at energies around 0.511 MeV, which corresponds to their rest mass energy. However, according to special relativity, they can never reach the speed of light, no matter how much energy they gain. Protons become relativistic at much higher energies, near 938 MeV, their rest mass energy. Similarly, heavy ions, they require even higher energies (in the GeV per nucleon range) to exhibit relativistic effects due to their much greater mass. Regardless of particle type, no material particle can attain the speed of light; they can only asymptotically approach it as their energy increases. This is illustrated in Figure 1.1.

To maintain particles in a stable orbit as a beam, they must be placed in a constant magnetic field, where they spiral around the magnetic lines of force. This leads to the concept of magnetic rigidity, defined alongside the magnetic force, cyclotron frequency, and gyroradius as follows:

Magnetic force on particle:

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B}) \tag{1.7}$$

Cyclotron frequency of the particle:

$$\omega_c = \frac{qB}{m} \tag{1.8}$$

Larmor radius (gyroradius) of particle:

$$\rho = \frac{mv}{qB} \tag{1.9}$$

Magnetic rigidity:

$$B\rho = \frac{p}{q} = \frac{mv}{q} \tag{1.10}$$

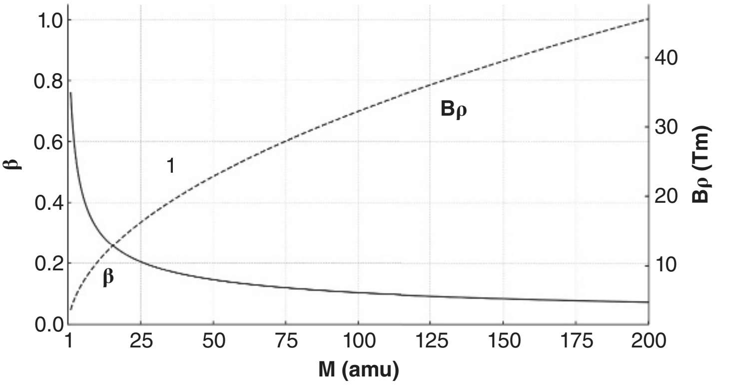

Figure 1.2 Velocity and magnetic rigidity of different ion beams of mass M (amu) at 500 MeV energy.

In terms of practical units:

$$B\rho\,(\mathrm{Tm}) = 3.3356 \cdot \frac{p\,(\mathrm{GeV}/c)}{q} \qquad \text{(relativistic)}$$
$$B\rho\,(\mathrm{Tm}) = 0.1439 \cdot \sqrt{\frac{M\,(\mathrm{amu}) \cdot E\,(\mathrm{MeV})}{q^2}} \qquad \text{(non-relativistic)} \tag{1.11}$$

Here, E is the total energy of the particle. Using Equation 1.11, if we plot the magnetic rigidity required to bend different ions with a fixed energy of 500 MeV and a unit charge state, we see that it increases with the mass of the ions. This is shown in Figure 1.2, alongside the corresponding normalized particle velocities. Following conclusions can be drawn:

1. Maximum magnetic rigidity is calculated for the heaviest mass, highest energy, and unit positive charge state of the beam desired in the accelerator.
2. Magnetic rigidity increases with mass and energy of charged particles for a given charge state of the beam particles.
3. Magnetic rigidity is maximum for a unit charge state of an ion beam with target energy.
4. Magnetic rigidity depends on the mass-to-charge (m/q) ratio directly, When $E = qV$ where V is the voltage by which particles are accelerated.

In 1909, Ernest Rutherford bombarded alpha particles onto a thin gold foil. To overcome the Coulomb barrier between the alpha particles and the gold nucleus, high energies were necessary to surmount the Coulomb repulsion. The higher the energy imparted to particles, the shorter their de Broglie wavelength (λ). Just as a living cell is observed under an optical microscope using scattered visible light photons, energetic particles can be used to probe matter, depending on their wavelength. The wavelength of energetic particles determines the size of the object to be resolved, so high-energy particles (with mass m, velocity v, and energy E) are required to probe deep into atoms and nuclei. The de Broglie wavelength (λ) is given by the Planck–Einstein relation, the relationship between energy and momentum:

$$E = h\nu = \frac{hc}{\lambda} = pc \tag{1.12}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\sqrt{2mE}} \quad (1.13)$$

In terms of practical units:

$$\lambda(\mathring{A}) = \frac{12.27}{E(eV)} \quad (1.14)$$

Here, E is the energy of the particle (or photon), h is Planck's constant, ν is the frequency of the de Broglie wave associated with the particle.

Accelerators have evolved over the last two centuries as a result of applying electricity and magnetism to charged particles, driven by the pioneering work of many great scientists and engineers. Initially motivated by Rutherford's famous experiment to explore the nucleus, modern particle accelerators are essential tools for probing the structure of atoms, protons, neutrons, and electrons through high-energy collisions that reveal their internal components. They enable the discovery of new particles, such as quarks, leptons, and the Higgs boson, by recreating the extreme conditions necessary for these short-lived particles to manifest. Studying particle interactions at high energies allows scientists to explore the fundamental forces of nature – gravity, electromagnetism, and the strong and weak nuclear forces – and how they behave and unify. Additionally, accelerators like the Large Hadron Collider (LHC) recreate the energy densities present just after the Big Bang, offering insights into the origins of the universe. They also provide a platform to test and confirm theoretical models of quantum mechanics and relativity, including the Standard Model, by observing particle behavior and interactions with exceptional precision.

Thus, the evolution of accelerators has progressed from natural particle accelerators like cosmic rays and radioactive materials to highly engineered devices capable of producing extreme energetic particles. Cosmic rays, consisting of high-energy particles originating from astrophysical events, represent the earliest and most powerful natural accelerators. Similarly, radioactive sources emit high-energy particles during nuclear decay, naturally accelerating alpha, beta, and gamma particles. The first major technological breakthrough in particle acceleration came with the development of the cathode ray tube (CRT), which used high voltage to accelerate electrons and produce visible images, laying the groundwork for more sophisticated human-made accelerators.

In the early twentieth century, the invention of the Van de Graaff accelerator [2] marked a significant advancement in accelerating charged particles like protons and ions using high voltage generated by a moving belt. This innovation paved the way for circular accelerators like the cyclotron, where charged particles like protons and deuterons are accelerated in a magnetic field and gain energy from an alternating electric field. For electrons, which encounter relativistic effects at higher speeds, the betatron was developed, using a time-varying magnetic field to induce electron acceleration in a circular orbit.

As accelerators grew more powerful, the synchrotron was introduced in the mid-twentieth century, where protons, electrons, and ions are accelerated in a circular path with synchronized magnetic and electric fields, achieving energies in the TeV range. Synchrotrons facilitated the discovery of particles like quarks and leptons and contributed significantly to particle physics. Around the same time, the linear accelerator (LINAC) emerged, allowing particles to be accelerated in a straight path using oscillating electric fields. A variant of the Van de Graaff, the Tandem accelerator, was designed to stabilize high-voltage acceleration using a chain of metal pellets to achieve greater reliability and higher energies.

Following these advancements, the microtron was created to reuse circular orbits for electrons, allowing them to gain energy with each pass. In the mid-twentieth century, the fixed-field alternating gradient (FFAG) accelerator was introduced, accelerating particles in a spiral path while maintaining a constant magnetic field. The nonscaling FFAG improved upon this design by allowing more rapid acceleration with variable particle orbits. Meanwhile, the induction LINAC utilized pulsed magnetic fields to accelerate particles like electrons and ions over shorter distances.

In the era of modern high-energy physics, particle colliders like the LHC emerged, accelerating two beams of particles in opposite directions to create collisions at extreme energies, often in the TeV range. Further innovations included the recirculating LINAC, which allowed particles to pass through the same LINAC multiple times to reach higher energies, as well as the energy recovery LINAC, which focuses on energy efficiency by decelerating electron beams and recovering their energy for reuse.

Most recently, breakthroughs in compact acceleration methods have led to the development of laser plasma accelerators, which use high-intensity lasers to generate plasma wakefields, rapidly accelerating particles like electrons to GeV energies. Beam-driven plasma accelerators follow a similar principle, using a high-energy particle beam to generate a plasma wakefield, offering compact and efficient acceleration over shorter distances.

Plasma wakefield accelerators, driven by either high-intensity lasers or particle beams, can achieve acceleration gradients in the order of gigavolts per meter (GV/m), far exceeding the capabilities of traditional accelerators. This allows for compact designs that can potentially reduce the size and cost of future particle accelerators.

1.1 History of Particle Accelerators and the Chronological Milestones

Before going into modern history accelerators, let's go back to one of the earliest known concepts of the atom in ancient Indian science. Maharishi Kanaad, an ancient Indian sage, was one of the first to propose the idea of atomic theory. He suggested that everything in the universe is made of tiny, indivisible particles called "parmanu," which are similar to what we now think of as atoms. He believed that these particles combine in various ways to form the objects and substances that we see around us. His ideas are recorded in the Vaisheshika Sutras, an important text in Indian philosophy. This was an early understanding that matter is made of basic building blocks.

Particle accelerators began to develop in the late nineteenth century, following important discoveries like X-rays by Wilhelm Röntgen and beta rays by J. J. Thomson. The discovery of radioactivity motivated scientists to find ways to create and control particles artificially. However, it was Ernest Rutherford's groundbreaking experiments that gave scientists the roadmap to study the atomic nucleus in more detail. The following is the timeline of key discoveries and advancements that shaped the history of particle accelerator technology and its pioneer discoveries as shown in Table 1.1. A brief history of accelerators and related discoveries is provided in Table 1.1 chronologically.

Table 1.1 Chronological evolution of particle accelerators worldwide.

Year	Developments
1895	Wilhelm Conrad Röntgen discovered X-rays using CRTs, for which he received the first Nobel Prize in Physics in 1901.
1896	Joseph John Thomson studied the nature of cathode rays, discovering their charge-to-mass ratio and identifying them as electron beams. He was awarded the Nobel Prize in 1906.
1898	Marie Curie discovered groundbreaking work in the study of radioactivity. In 1898, together with her husband Pierre Curie, she discovered two new elements: *polonium* in July and *radium* in December. She also introduced the term "radioactivity" to describe how certain materials give off energy as they break down. Her important discoveries rewarded her two Nobel Prizes: the first in Physics in 1903, shared with Pierre Curie and Henri Becquerel, for their research on radioactivity, and the second in Chemistry in 1911 for discovering radium, polonium, and studying the properties of radium. Marie Curie was the first woman to win a Nobel Prize and is still the only person to have won Nobel Prizes in two different sciences.
1920	John Douglas Cockcroft and Ernest Thomas Sinton Walton conceived the first high-voltage particle accelerator, which used two electrodes inside a vacuum vessel to create a potential drop of around 100 kV.
1923	Rolf Widerøe, a young Norwegian student, first proposed the design of the betatron, incorporating the well-known 2-to-1 rule. Two years later, he refined the design by introducing the condition for radial stability, a crucial factor for successful particle acceleration.
1924	Gustav Ising proposed time-varying fields across drift tubes, introducing the concept of resonant acceleration, which allows for energy levels higher than the system's maximum voltage.
1928–1932	John Cockcroft and Ernest Walton began designing an 800 kV generator under the guidance of Ernest Rutherford in 1928 at the Cavendish Laboratory in Cambridge. In 1932, they achieved a significant milestone by successfully splitting the lithium atom using high-energy protons accelerated by their Cockcroft–Walton generator. This marked the first artificial nuclear reaction, demonstrating the power of particle accelerators in nuclear physics. Their breakthrough experiment laid the foundation for future accelerator technologies and earned them the Nobel Prize in Physics in 1951 for their pioneering work in atomic transmutation.
1928	Rolf Widerøe demonstrated Ising's principle with a 1 MHz, 25 kV oscillator to produce 50 keV potassium ions, suggesting the use of RF voltage between consecutive drift tubes. This was successfully tested by D. H. Sloan and E. O. Lawrence in 1931.
1929	Ernest Lawrence was motivated by Rolf Widerøe's advances in accelerating particles and Gustav Ising's concept of using electric fields for this purpose. In 1929, Lawrence invented the cyclotron, a device that uses magnetic fields to move particles in a circular path while speeding them up with an electric field.
1930	M. Stanley Livingston, a student of Ernest Lawrence, built and tested the first working cyclotron. This machine accelerated hydrogen ions (protons) to an energy of 80 keV, marking a major step forward in particle accelerator technology.

(*Continued*)

Table 1.1 (Continued)

Year	Developments
1930	Robert Van de Graaff developed the Van de Graaff generator, an important type of electrostatic accelerator that allowed for the creation of extremely high voltages. This innovation was a critical step in early particle acceleration, enabling experiments that required high-energy beams.
1932	Lawrence's cyclotron accelerated protons to 1.25 MeV, successfully achieving atomic splitting shortly after Cockcroft and Walton's breakthrough. For this pioneering achievement, along with his invention and development of the cyclotron and his contributions to artificial radioactive elements, E. O. Lawrence was awarded the Nobel Prize in Physics in 1939.
1940	Donald W. Kerst revolutionized particle acceleration by redesigning the betatron and constructing the first fully functional machine capable of accelerating electrons to 2.2 MeV.
1944	Vladimir Veksler introduced the principle of phase stability, which made it possible to maintain stable orbits while increasing the energy of particles in synchrotrons. This discovery was vital for the development of modern particle accelerators such as synchrocyclotrons and electron synchrotrons.
1946	Luis W. Alvarez built a 32 MeV proton drift tube LINAC at Berkeley, which operated at 200 MHz, marking an important step in the development of LINACs.
1950	Donald W. Kerst constructed the largest betatron in the world, capable of accelerating electrons to 300 MeV, a record at the time.
1950	Nicholas Christofilos solved the challenge of beam stability in synchrotrons by introducing the concept of strong focusing. This advancement allowed synchrotrons to handle higher energy beams and led to more efficient acceleration in circular accelerators.
1951	J. D. Cockcroft and E. T. S. Walton were awarded the Nobel Prize for their pioneering work on the transmutation of atomic nuclei using artificially accelerated particles.
1951	Edwin M. MacMillan, along with Glenn T. Seaborg, was awarded the Nobel Prize for their research on the chemistry of transuranium elements.
1952	E. Courant, M. Livingston, and H. Snyder at Brookhaven, USA, proposed alternating-gradient (AG) magnetic lattices, consisting of bending magnets with AGs or properly spaced quadrupole magnets with alternating polarities. This innovation solved the weak focusing problem in synchrotrons.
1959	Emilio Segrè and Owen Chamberlain received the Nobel Prize in Physics for their discovery of the antiproton, achieved using a particle accelerator at the Lawrence Berkeley National Laboratory.
1960	Bruno Touschek designed the first practical electron–positron collider, which made it possible to study particle collisions in a controlled environment. His work laid the foundation for later developments in colliders like the large electron–positron collider.

Year	Developments
1965	Julian Schwinger, along with Sin-Itiro Tomonaga and Richard P. Feynman, was awarded the Nobel Prize for quantum electrodynamics, which has profound implications for elementary particle physics.
1966	Andrey M. Budker developed the technique of electron cooling, which enhanced beam quality in storage rings and improved the precision of particle accelerator experiments.
1968	Luis W. Alvarez received the Nobel Prize for discovering a large number of resonance states using a hydrogen bubble chamber.
1970	I. M. Kapchinsky and Vladimir Teplyakov introduced the concept of the radio frequency quadrupole in 1970. This innovation improved the acceleration and focusing of low-energy ion beams, leading to more efficient LINACs.
1976	Burton Richter and Samuel Ting were awarded the Nobel Prize in Physics for the independent discovery of the J/Ψ particle, providing evidence for the existence of the charm quark, discovered using high-energy particle accelerators.
1977	Rosalyn Yalow was awarded the Nobel Prize in Physiology or Medicine for developing the radioimmunoassay (RIA), a groundbreaking technique for measuring biological substances using radioactive isotopes. While radioisotopes used in RIA can be produced by particle accelerators, the work was primarily linked to medical diagnostics.
1980	Simon van der Meer invented stochastic cooling, a technique used to refine and manipulate particle beams in high-energy accelerators. This breakthrough was key to achieving the precision needed for particle collisions at the European Organization for Nuclear Research (CERN), which led to the discovery of the W and Z bosons.
1984	Carlo Rubbia and Simon Van der Meer were awarded the Nobel Prize for their contributions to the large project that led to the discovery of the W and Z bosons, the carriers of the weak force.
2013	François Englert and Peter Higgs were awarded the Nobel Prize in Physics for the theoretical discovery of the Higgs mechanism, confirmed by the discovery of the Higgs boson at the LHC at CERN.

1.2 Maxwell's Equations

In accelerator physics, the primary goal is often to solve Maxwell's equations for specific geometries to understand and optimize the electromagnetic fields in an accelerator components, such as radio frequency (RF) cavities, magnets, and electric components. It helps physicists and engineers to design accelerators with the desired beam dynamics, particle focusing, and energy transfer characteristics.

Consider an electromagnetic field moving through free space, which is an area with no electric charges or currents. The electric field, represented by $\mathbf{E}$, and the magnetic field, represented by $\mathbf{B}$, change over time and space and affect each other. Maxwell's equations describe how these fields behave and interact. The symbol ρ stands for charge density, which is the amount of charge in a given space. The term J is the current density, indicating the flow of electric charge. Now, here are Maxwell's equations in both integral and differential forms:

1.2.1 Maxwell's First Equation: Gauss's Law for Electricity

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{(Differential form)} \tag{1.15}$$

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enc}}}{\varepsilon_0} \quad \text{(Integral form)} \tag{1.16}$$

This law states that the divergence of the electric field ($\mathbf{E}$) at any point in space is proportional to the local charge density (ρ). It describes how electric fields emanate from electric charges. In accelerators, electric fields play a key role in accelerating and focusing on charged particle beams. Longitudinal electric fields provide electrostatic acceleration, while transverse electric fields, particularly in electrostatic quadrupoles, focus the beam. Dipole configurations use electric fields for deflecting or steering particles. Gauss's law underpins the theoretical framework for understanding how electric fields interact with charges in accelerator environments.

1.2.2 Maxwell's Second Equation: Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \text{(Differential form)} \tag{1.17}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \quad \text{(Integral form)} \tag{1.18}$$

This law states that the divergence of the magnetic field ($\mathbf{B}$) is zero, implying the absence of magnetic monopoles. Magnetic field lines always form closed loops. In accelerators like cyclotrons, synchrotrons, or betatrons, magnetic fields are essential for bending and focusing on charged particles, following this fundamental property.

1.2.3 Maxwell's Third Equation: Faraday's Law of Electromagnetic Induction

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{(Differential form)} \tag{1.19}$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{A} \quad \text{(Integral form)} \tag{1.20}$$

Faraday's law describes how a changing magnetic field induces an electric field. In accelerators like the betatron, this principle is employed to induce electric fields that accelerate particles. A time-varying magnetic field generates a circulating electric field, which accelerates electrons or other particles confined within a circular path. This law is fundamental to devices using electromagnetic induction for acceleration.

1.2.4 Maxwell's Fourth Equation: Ampère's Law with Maxwell's Addition

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad \text{(Differential form)} \tag{1.21}$$

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \varepsilon_0 \frac{d}{dt} \int_S \mathbf{E} \cdot d\mathbf{A} \quad \text{(Integral form)} \tag{1.22}$$

Ampère's law, with Maxwell's correction, incorporates the displacement current ($\frac{\partial \mathbf{E}}{\partial t}$) alongside the conduction current density (**J**). This law describes how electric currents and changing electric fields generate magnetic fields. In accelerator design, this equation is crucial for calculating ampere-turns in dipoles, quadrupoles, sextupoles, and solenoids. It also helps in the design of power supplies for energizing magnets in accelerators.

1.3 Electrostatic Accelerators

As the name suggests, electrostatic accelerators use electric fields to accelerate charged particles. The design of the electric field is crucial, as these accelerators are characterized by a potential gap where a particle experiences a potential difference and accelerates along the direction of the electric field. The voltage can reach very high levels (up to MV) with the use of suitable insulating gases like SF_6. Examples of electrostatic accelerators include Cockcroft–Walton accelerators, Van de Graaff accelerators, and tandem accelerators.

When a charge q moves through an electric field from one point to another, work is done by the electric field. The work done W is given by the change in electric potential energy, which can be expressed as:

$$W = q(V_1 - V_2)$$

where V_1 and V_2 are the electric potentials at the two points.

If we consider the charge moving from a point of zero potential (grounded) to a point with potential V, the work done by the field is:

$$W = qV$$

This work is the energy gained by the charge, so the energy gained by a particle with charge q under a voltage V is simply given by:

$$E = qV$$

The force (F) on a charge particle in an electric field (**E**) is given by:

$$\mathbf{F} = q\mathbf{E}$$

1.3.1 Cockcroft–Walton Accelerator

The Cockcroft–Walton accelerator, first developed in 1932 by John Cockcroft and Ernest Walton [3, 4], played a significant role in nuclear physics. It operates on the principle of cascading voltage multiplication. The system works by charging and discharging capacitors in cascading stages, with the voltage increasing stepwise at each stage. The circuit consists of a

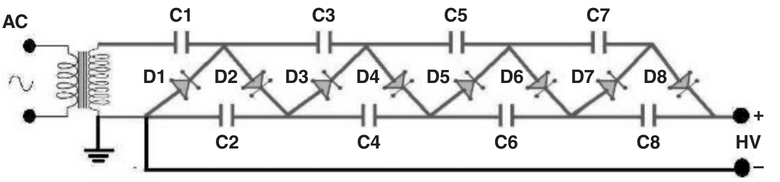

Figure 1.3 Schematic of a Cockcroft–Walton accelerator.

series of capacitors and diodes arranged in a specific configuration to create a voltage multiplier. The output voltage (V_{out}) increases with the number of stages (N), the input voltage (V_{in}), and the capacitance of the capacitors, although size and breakdown limits impose certain constraints. In the setup as shown in Figure 1.3, an AC voltage charges the capacitors in parallel so that each stage generates a potential twice the input voltage. The first capacitor charges in an anticlockwise loop, and the second in a clockwise loop, followed by the discharge of the first capacitor. With more stages, the final voltage increases until it reaches the breakdown voltage, as shown in Figure 1.3, where one can stop further cascading.

In a four-stage Cockcroft–Walton voltage multiplier circuit as shown in Figure 1.3, the circuit consists of two columns of capacitors: the oscillating column and the smoothing column. During the first half-cycle of the AC input, the capacitors in the oscillating column (C_1, C_3, C_5, and C_7) are charged through the odd-numbered diodes (D_1, D_3, D_5, and D_7). During the next half-cycle, the capacitors in the smoothing column (C_2, C_4, C_6, and C_8) are charged through the even-numbered diodes (D_2, D_4, D_6, and D_8). In the steady state, under no-load conditions, each capacitor in the smoothing column is charged to a voltage of $2V_{\mathrm{m}}$, which is twice the peak input voltage.

If the generator supplies any load current I, the output voltage will not reach the theoretical value of $2nV_{\mathrm{m}}$, as depicted in Figure 1.4. Additionally, a ripple is introduced in the voltage, necessitating the consideration of both the voltage drop ΔV_0 and the peak-to-peak ripple $2\delta V$.

The maximum output voltage V_{out} for an n-stage Cockcroft–Walton multiplier is expressed as:

$$V_{\mathrm{out}} = n \cdot 2V_{\mathrm{m}} \tag{1.23}$$

where n represents the number of stages, and V_{m} denotes the peak input voltage.

For a 4-stage multiplier, the maximum output voltage equals eight times the peak input voltage:

$$V_{\mathrm{out}} = 4 \times 2 \cdot V_{\mathrm{m}} = 8V_{\mathrm{m}} \tag{1.24}$$

Let q denote the charge transferred from capacitor C_n to the load in each cycle. The ripple voltage across capacitor C_n is given by:

$$2\delta V = q \sum_{1}^{n} \frac{1}{C_n} \tag{1.25}$$

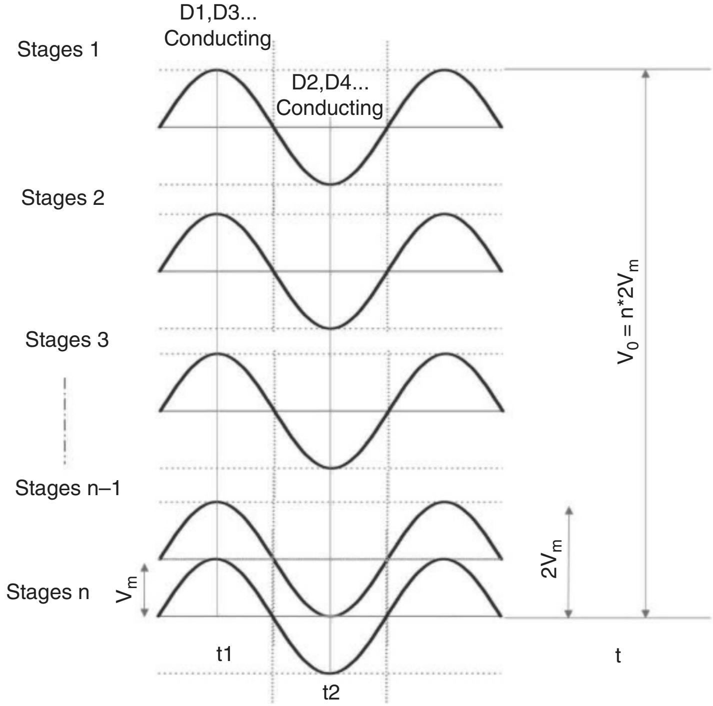

Figure 1.4 Voltage flow in Cockcroft–Walton multiplier circuit.

Assuming that all smoothing capacitors have equal capacitance *C*, and considering that the ripples across all capacitors are in phase, the total ripple voltage on the load is:

$$2\delta V = q\frac{n(n+1)}{2C} = qI\frac{n(n+1)}{2f} \tag{1.26}$$

For equal capacitances in the cascade circuit, the voltage drops across individual stages are:

$$\Delta V_n = \frac{q}{C_n}, \tag{1.27}$$

$$\Delta V_{n-1} = \frac{q}{C}[2n-(n-1)], \tag{1.28}$$

$$\Delta V_1 = \frac{q}{C}[2n+2(n-1)+2(n-2)+\cdots+2(2)+1]. \tag{1.29}$$

By summing and substituting $q = \frac{I}{f}$, the total voltage drop is:

$$\Delta V_0 = \frac{I}{fC}\left(\frac{2n^3}{3}+\frac{n^2}{2}-\frac{n}{6}\right) \tag{1.30}$$

The lower capacitors primarily contribute to ΔV_0, similar to the ripple effect. Doubling C_0 is advantageous as it only needs to withstand half the voltage of other capacitors, i.e. $V_{\max}$ Consequently, ΔV_n reduces by, decreasing ΔV_0 of each stage by the same factor, n-times:

$$\Delta V_0 = \frac{I}{fC}\left(\frac{2n^3}{3} - \frac{n}{6}\right). \tag{1.31}$$

For $n \geq 4$, the linear term becomes negligible, and the maximum output voltage is approximated as:

$$V_0^{\max} \approx 2nV_m - \frac{I}{fC} \cdot \frac{2n^3}{3} \tag{1.32}$$

For a fixed number of stages, if ΔV is the voltage drop due to loading effects then the average output voltage is given as follows:

$$V_0 = V_0^{\max} - \Delta V \tag{1.33}$$

This voltage decreases linearly with load current I at constant frequency. Initially, V_0 rises with the number of stages n, but beyond an optimum stage count, it declines. The optimum number of stages that yields the maximum usable output voltage under load is, determined by differentiating Equation (1.32) with respect to n, is:

$$n_{\text{opt}} = \sqrt{\frac{V_m fC}{I}} \tag{1.34}$$

This expression serves as a design criterion for practical CW multipliers to ensure high voltage efficiency without excessive stage losses.

1.3.2 Tandem Accelerators

Tandem accelerators are a type of electrostatic accelerator, similar to the Van de Graaff accelerator, but they utilize the same high-voltage twice. They are also known as Pelletron accelerators [5] because they use metallic pellets to carry charge on a belt, building up a large megavolt potential. A substantial electrostatic potential (V_T) is created at the center of a large insulating tank, usually filled with SF_6 gas, and distributed across various accelerating tubes to ensure linear and smooth acceleration. Tandem accelerators can be configured horizontally or vertically, depending on space constraints. The schematic is shown in Figure 1.5.

Here is the sequence of operation for a non-relativistic particle:

1. The charging chain or belt is given an electrical charge at a lower potential and is moved mechanically towards the high-voltage terminal.
2. The charge is transferred from the chain to the high-voltage terminal, causing the terminal's voltage to increase.
3. The terminal reaches a very high voltage V_T, usually in the range of several million volts, generating a strong electrostatic field. It is utilized linearly by distributing over many accelerating tubes through a set of resistor networks.
4. A negative ion source at the top of the accelerator tank as shown in Figure 1.5 produces negatively charged particles (ions), which are then injected at energy E_{Inj} into the

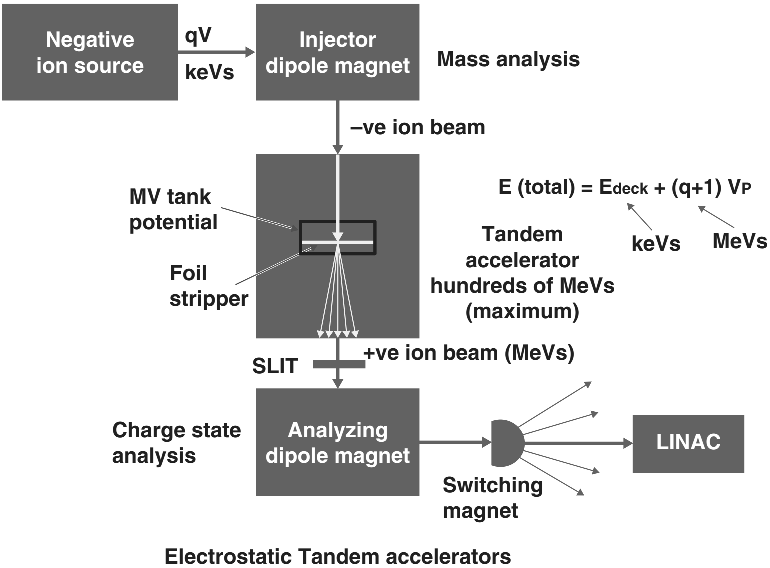

Figure 1.5 Schematic of a Tandem accelerator.

accelerator tank containing many accelerating tubes in series. It is done by an injector dipole magnet, which not only defines the energy of the beam but also selects the particular mass of the beam in one unit negative charge state by the following non-relativistic formula as follows:

$$B_{inj}\rho_{inj}(Tm) = 0.1439\sqrt{\frac{M(amu)E_{inj}(MeV)}{(-1)^2}} \tag{1.35}$$

where ρ_{inj} is the bending radius of the injector magnet. Here, the injector magnet is usually mass selector for the beam under a given acceleration (deck) potential. The main accelerator high voltage (MV) (V_T) is used twice in action, first attracting the negative ion beam from the top of accelerator tank and then stripping the negative ion into a positive ion and hence repelling it by the same high voltage using either a gas or foil stripper.

5. The ions of various charge states after the stripping are accelerated through the tube towards the ground potential, gaining kinetic energy (E_T) by a particular charge state ion (q) as they move as per formula:

$$E_T = E_{Inj} + (q + 1)V_T \tag{1.36}$$

6. The accelerated ion beam is then analyzed in terms of energy by an analyzing dipole magnet and the beam is transported to an experimental area with a defined energy and a defined beam current. The corresponding magnetic field (B_{anl}) is given as per non-relativistic formula as follows:

$$B_{anl}\rho_{anl}(Tm) = 0.1439\sqrt{\frac{M(amu)E_T(MeV)}{(q)^2}} \tag{1.37}$$

where ρ_{anl} is the bending radius of analyzer magnet. Here, analyzer magnet is the energy and charge state selector of the beam.

7. In order to stabilize the beam, a dynamic feedback of beam current is taken from analyzer magnet image slits (known as high and low energy slits in bending plane of analyzer magnet), which is fed back for main stabilization.
8. After successful acceleration, beam is transported in beamline. A switching dipole magnet having many output beamlines is used such that for a given magnetic field in the switching magnet, beam can be diverted into one of the beamlines. The corresponding magnetic field (B_{sw}) is given as per non-relativistic formula as follows:

$$B_{sw}\rho_{sw}(Tm) = 0.1439\sqrt{\frac{M(amu)E_T(MeV)}{(q)^2}} \tag{1.38}$$

where ρ_{sw} is the bending radius of the switcher magnet for a particular beamline out of various output ports. Here, a switching magnet is the path selector for beam.

The maximum voltage at the terminal [6] of a Tandem accelerator depends on the size of the terminal electrode, which functions like a spherical capacitor. The capacitance C of this spherical capacitor is given by the formula:

$$C = 4\pi\epsilon_0 r \tag{1.39}$$

which in practical unit becomes:

$$C\,(\mathrm{F}) = 1.11 \times 10^{-10} \cdot r\,(\mathrm{m}) \tag{1.40}$$

where r is the radius of the terminal electrode.

If the spherical terminal with radius r_1 is surrounded by a grounded concentric shell with radius r_2, the capacitance C is calculated by:

$$C = 4\pi\epsilon_0 \frac{r_1 r_2}{r_2 - r_1} = (1.11 \times 10^{-10}\ \mathrm{F/m})\frac{r_1 r_2}{r_2 - r_1} \tag{1.41}$$

The terminal voltage V depends on the charge Q stored on the terminal electrode and is expressed by:

$$V = \frac{Q}{C} \tag{1.42}$$

The maximum voltage that can be achieved at equilibrium, denoted as V_0, is influenced by several currents: the current I_{belt} carried by the belt, the beam current I_{b} in the accelerating tube, the current I_{r} through the resistor chain (which includes the current due to the remaining resistance of the insulating column), and the corona current I_{c}, which rises sharply after a certain point. The time evolution is expressed as follows:

$$\frac{dV}{dt} = \frac{1}{C}\frac{dQ}{dt} = \frac{1}{C}(I_{\mathrm{belt}} - I_{\mathrm{b}} - I_{\mathrm{c}} - I_{\mathrm{res}}) \tag{1.43}$$

When the terminal voltage is low, and the corona current I_{c} is small, this equation can be solved more easily. By introducing the effective resistance R of the resistor chain and the insulating column, the equation simplifies to:

$$\frac{dV}{dt} = \frac{1}{C}\left(I_{\mathrm{belt}} - I_{\mathrm{b}} - \frac{V}{R}\right) \tag{1.44}$$

Here, R represents the effective resistance of the resistor chain and insulating column, measured in ohms (Ω). This resistance plays a crucial role in controlling the voltage drop and charge dissipation in the system. The corona current I_{cor} increases sharply beyond a threshold terminal voltage, typically when the electric field strength exceeds the breakdown strength [7] of the surrounding medium. This threshold depends on factors such as the geometry of the terminal, the type of insulating gas used (e.g. SF_6, and the pressure of the gas. For Tandem accelerators, the terminal radius r_1 typically ranges from tens of centimeters to a few meters, and the equilibrium voltage V_0 can reach several million volts, depending on the application. The exponential term $e^{-t/\tau}$ in the solution indicates how quickly the terminal voltage stabilizes, with the time constant $\tau = \text{RC}$ governing this rate; larger τ results in slower stabilization. The fast feedback control system continuously monitors parameters, such as terminal voltage and beam energy spread. By adjusting the charging current I_{belt} or corona load I_c, it ensures stability within a precision range of typically less than 0.01% energy spread. Maintaining a highly stable terminal voltage is essential for minimizing beam energy fluctuations, which directly affect the resolution and accuracy of experiments relying on the Tandem accelerator.

The time constant $\tau = RC$ and the equilibrium voltage V_0 are defined as:

$$V_0 = R\,(I_{belt} - I_b) \tag{1.45}$$

The solution to the differential equation is:

$$V(t) = V_0 + [V(0) - V_0]\,e^{-t/\tau} \tag{1.46}$$

This solution is valid as long as the currents I_{belt}, I_b, and the resistance R remain constant, and the corona current is negligible. The terminal reaches an equilibrium voltage V_0 when the charge supplied by the belt equals the charge leaving the terminal. In Tandem accelerators, fast feedback control systems usually by analyzing magnet slit currents from beam are used to quickly adjust and stabilize the terminal voltage V. The behavior of V over time is determined by the characteristics of this control system. The terminal voltage should not drift over time and must be highly stabilized to get the beam with a very small energy spread. The terminal voltage can be managed by regulating the charging current I_{belt} and adjusting the corona load current using movable needles.

Regarding strippers, gas and foil strippers are two commonly used methods for changing the charge state of ions in accelerators, each with its advantages and disadvantages. Gas strippers involve passing the ion beam through a low-density gas, where collisions with gas molecules strip electrons from the ions. They generally result in lower charge states and less efficient stripping compared to foil strippers. On the other hand, foil strippers use thin metal foils that the ion beam passes through, where the high-density material more aggressively strips electrons, often leading to higher charge states and more efficient stripping. While foil strippers can produce higher charge states, they tend to cause more scattering and energy loss in the ion beam, which can degrade the beam quality. The choice between gas and foil strippers depends on the specific requirements of the accelerator, such as the desired charge state and beam quality.

The first charge state (either gas or foil) in the Tandem accelerator is usually inside the terminal voltage tank where polarity reversal of ion takes place. Sometimes, one may use two strippers in Tandem accelerator, which makes q1 and q2 charge state of ions so as to get the high charge state q2 of ions. For example, if there is a total n stage of acceleration, the second stripper lies at kth section, then we have total energy from Tandem accelerator as follows:

The total energy E_T of an ion in a Tandem accelerator with a double stripper configuration can be calculated using the following formula:

$$E_T = E_{\text{inj}} + V\left(1 + \frac{q_1 \cdot k}{n} + \frac{q_2 \cdot (n-k)}{n}\right) \tag{1.47}$$

Tandem accelerators, especially Tandem accelerators, are an important advancement in electrostatic acceleration technology. They work by using the same high-voltage potential twice to achieve higher energy for particles. These accelerators are used in many areas, including nuclear physics, material science, and medical treatments. The ability to use both gas and foil strippers allows for flexible control of the particle charge states, helping to improve beam efficiency while reducing energy loss. The tandem design combines sequential electrostatic fields by accelerating tubes to accelerate particles through multiple stages. Magnets such as injector, analyzer, and switching magnets are essential in determining the mass, energy, and direction of the beam. In addition, feedback systems ensure that the terminal voltage remains stable, which helps produce a beam with minimal energy spread.

1.4 Cyclotrons

The concept of the cyclotron [8, 9] was introduced by E. O. Lawrence in 1929. Within a few years, Lawrence, along with Edlefsen, constructed the first circular particle accelerator: a 4-inch cyclotron, and the successful acceleration of particles was demonstrated by M. S. Livingston. This innovation marked a significant advancement, as it was the first accelerator capable of delivering multiple accelerating impulses to a particle beam using a single pair of electrodes. Ernest Lawrence was awarded the Nobel Prize in Physics in 1939 for this invention. Cyclotrons accelerate particles in a spiral trajectory under a homogeneous magnetic field, with acceleration provided by a time-varying electric field.

Lawrence's pioneering work led to significant milestones, such as the successful acceleration of hydrogen ions to 80 keV in 1931 and achieving 1.25 MeV proton acceleration in 1932, shortly after the achievements of Cockcroft and Walton. The cyclotron operates in a high-vacuum environment, typically around 10^{-6} torr, which is essential to minimize beam loss due to collisions with residual gas molecules and to ensure stable high-voltage operation of the dees and deflector. The acceleration process relies on the repeated use of the same electrodes, where charged particles gain energy twice per revolution as they cross the gap between the dees, gradually increasing their spiral orbit radius. The motion of charged particles within the cyclotron occurs in the median plane, where the applied magnetic field guides them in near-circular orbits.

Repeated acceleration requires resonance between the particle's revolution frequency and the applied RF. The resonance condition must be maintained throughout the acceleration process to ensure the particles reach the accelerating gaps at the correct time. Any deviation between the particle revolution frequency and the RF results in a phase shift between the beam particles and the driving RF field, causing the particles to receive less energy. The acceleration process reaches its limit when the phase shift approaches ±90°. Beyond this point, particles experience deceleration and are eventually lost from the system.

The particles are subjected into circular motion using homogeneous magnetic field. As the energy of the particles increases, so does the orbit radius, but if the charge (q), magnetic field (B), and mass (m) remain constant, the frequency stays the same across all energies, velocities, and radii. By positioning two electrodes (known as dees) around an ion source, ions can gain energy twice per revolution, consistently arriving at the same azimuth for each subsequent acceleration. An accelerator that maintains a constant orbital frequency is termed isochronous cyclotron. Positive ions are extracted from the ion source whenever the voltage on the first dee is negative. Each accelerating pulse increases the beam's orbital radius, creating a spiral trajectory. Near the edge of the dipole magnet pole, the beam is extracted using an electrostatic deflector. For a positively charged beam, a negatively charged outer electrode pulls the beam from its orbit, while a thin, grounded inner electrode, known as the septum, separates the exiting beam from the inner orbits.

A cyclotron is not suitable for accelerating electrons primarily because of their low mass, which causes them to quickly reach relativistic speeds even at relatively low energies. As electrons approach the speed of light, their mass effectively increases, leading to a decrease in their angular frequency. This change causes them to fall out of synchronization with the fixed-frequency alternating electric field used in the cyclotron. Thus, the fixed magnetic field in a cyclotron is unable to maintain the proper trajectory for relativistic electrons, making it difficult to keep them on the intended spiral path. These factors together prevent a cyclotron from efficiently accelerating electrons.

The cyclotron operates based on the principle of cyclotron resonance. For a particle of mass (m), charge (q), and velocity (v) moving in a static magnetic field (B) along a circular path of radius (R), the motion is governed by:

$$\frac{mv^2}{R} = qvB \tag{1.48}$$

$$v = \frac{qBR}{m} \tag{1.49}$$

The cyclotron frequency is:

$$f = \frac{v}{2\pi R} = \frac{qB}{2\pi m} \tag{1.50}$$

For relativistic case,

$$f_{\text{cyc, rel}}(\text{MHz}) = \frac{Q}{A} \cdot \frac{15.23 \cdot B(T)}{\gamma} \tag{1.51}$$

$$\gamma = 1 + \frac{E_{\text{kinetic}}}{A \cdot 938.27} \tag{1.52}$$

The kinetic energy of particles is given by:

$$E = \frac{1}{2}mv^2 = \frac{(qBR)^2}{2m} \tag{1.53}$$

Substituting $q = Qe$ and $m = A \cdot m_u$: where m_u is one unit of mass in amu.

Kinetic energy per nucleon:

$$\frac{E}{A} = \frac{(Qe \cdot B \cdot R)^2}{2A^2 \cdot m_u} \tag{1.54}$$

Cyclotron K-factor:

$$K = \frac{(eBR)^2}{2m_u} \tag{1.55}$$

$$\frac{E}{A} = K\left(\frac{Q}{A}\right)^2 \tag{1.56}$$

The K-factor is an important figure of merit for cyclotrons, defining the upper limit of the kinetic energy that can be achieved for proton given the cyclotron's magnetic field and maximum orbital radius. This expression serves as a figure of merit for a cyclotron, also referred to as the bending limit, and can be expressed for protons as:

$$K(\text{MeV}) = 47.9(BR(\text{Tm}))^2 \tag{1.57}$$

The weight of a cyclotron approximately increases with the K value as:

$$W \approx K^{1.5} \approx (BR)^3 \tag{1.58}$$

Finally, the time to complete one revolution is:

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} \tag{1.59}$$

Thus, the period of revolution is independent of the velocity and size of the orbit, depending only on the magnetic field and the charge-to-mass ratio of the particle. When these are constant, the period of rotation is constant. For resonance, the RF period must coincide with this period of revolution. The cyclotron frequency is given by:

$$\omega = \frac{2\pi}{T} = \frac{qB}{m} \tag{1.60}$$

According to the theory of relativity, as the energy or velocity of particles increases, so does their mass as they approach the speed of light. Consequently, the cyclotron frequency of fast particles becomes lower than that of slower particles, leading to a gradual loss of synchronism.

The energy gain (E_k) of a particle under the dee's potential (V) is:

$$E_k = qV \tag{1.61}$$

The particle must pass through the dees n times to reach the required energy (E):

$$E_{\max} = nE_k \tag{1.62}$$

Thus, the maximum radius of the cyclotron orbit is:

$$R_{\max} = \frac{\sqrt{2mE_{\max}}}{qB} \tag{1.63}$$

In summary, the fundamental principle of cyclotron resonance ensures that particles maintain a stable, circular orbit while continuously gaining energy from the electric field as they pass through the dees. The critical relationship between the magnetic field strength, particle charge, mass, and velocity allows the cyclotron to deliver high-energy particles in a compact design. The introduction of the cyclotron K-factor helps quantify the energy limits achievable within the cyclotron based on the magnetic field strength and orbital radius. However, cyclotrons face limitations when dealing with particles that reach relativistic speeds, such as electrons. As particles approach the speed of light, their relativistic mass increases, leading to a loss of synchronization with the RF field, which renders the cyclotron less effective for accelerating particles at such high velocities.

1.5 Synchrotrons

A synchrotron [10] is a circular ring accelerator that functions similarly to a cyclotron, but with a magnetic field that increases with the energy of the particles to maintain a constant circular radius. This variation, known as alternate gradient ensures that the particles remain in resonance with the accelerating RF cavities, which continue to boost energy with each revolution. Synchrotrons are particularly well-suited for accelerating electron beams, making them powerful sources of X-rays and ultraviolet (UV) light. The emitted photons are exceptionally bright, with a narrow emittance cone, and their wavelength is tunable, making them useful in various experiments and applications. Synchrotrons can also accelerate protons and heavy ions. The largest synchrotron-type accelerator in the world is the Large Hadron Collider (LHC), with a circumference of 27 kilometers, located at the European Organization for Nuclear Research (CERN) near Geneva, Switzerland. Other notable synchrotron facilities include the Advanced Photon Source (APS) at Argonne National Laboratory in the United States and the Diamond Light Source in the United Kingdom.

A storage ring is a specialized type of synchrotron where the particles maintain constant energy. The main component of a synchrotron is a ring-shaped vacuum vessel, where charged particles approach nearly the speed of light and are guided by powerful magnets. A synchrotron produces extremely intense beams of light, known as synchrotron radiation. This radiation is used in various disciplines, including physics, chemistry, biology, and materials science.

As particles gain energy in the synchrotron, they emit energy in the form of synchrotron radiation. The energy loss due to synchrotron radiation, especially for high-energy electrons, is significant and scales with the fourth power of the particle's energy. The power radiated due to synchrotron radiation is given by:

$$P_{\text{synch}} = \frac{e^2 c}{6\pi\epsilon_0} \frac{\gamma^4}{R^2} \tag{1.64}$$

where P_{synch} is the synchrotron radiation power, γ is the Lorentz factor, and R is the bending radius of the synchrotron. As the particle's energy increases, the radiated power becomes a limiting factor, requiring careful design considerations to balance energy loss and acceleration.

The alternating gradient (strong focusing) technique is another important feature of synchrotrons. This technique uses quadrupole magnets to focus the particle beam transversely, ensuring that the beam remains stable while being accelerated. The quadrupole magnets create alternating focusing and defocusing forces, increasing beam stability. The focusing strength K of a quadrupole magnet is given by:

$$K = \frac{1}{f} \tag{1.65}$$

where f is the focal length of the quadrupole magnet. This method of strong focusing allows for higher energy beams to be accelerated in more compact synchrotron designs, increasing the efficiency and stability of the accelerator.

For a synchrotron ring containing particles with energy E and N dipole magnets of effective length l, the magnetic rigidity of the bending magnet is given by:

$$B\rho = \frac{\beta E}{q} \tag{1.66}$$

The integrated dipole strength is given by:

$$B.l = B(\rho\theta) = \frac{\beta E}{q}\frac{2\pi}{N} \tag{1.67}$$

The filling factor K_F is defined as the ratio of the total bending trajectory length to the circumference C of the ring:

$$K_F = \frac{Nl}{C} \tag{1.68}$$

The ring circumference is given by:

$$C = \frac{Nl}{K_F} = \frac{2\pi\beta E}{K_F Bq} \tag{1.69}$$

The relationship between the RF $f_{\mathrm{RF}}(t)$ and the revolution frequency $f_{\mathrm{rev}}(t)$ in a synchrotron is given by the harmonic number h, such that:

$$f_{\mathrm{RF}}(t) = h \cdot f_{\mathrm{rev}}(t) \tag{1.70}$$

The revolution frequency $f_{\mathrm{rev}}(t)$ can be expressed as the velocity $v(t)$ of the particle divided by the circumference of the synchrotron $2\pi R_s$:

$$f_{\mathrm{rev}}(t) = \frac{v(t)}{2\pi R_s} \tag{1.71}$$

The velocity $v(t)$ of the particle can be related to its momentum $p(t)$ and total energy $E(t)$ using the equation:

$$v(t) = \frac{p(t) \cdot c^2}{E(t)} \tag{1.72}$$

The momentum $p(t)$ of a charged particle moving in a magnetic field $B(t)$ with a bending radius ρ is given by:

$$p(t) = e \cdot B(t) \cdot \rho \tag{1.73}$$

The total energy $E(t)$ of the particle, considering relativistic effects, is expressed as:

$$E(t) = \sqrt{(m_0c^2)^2 + (p(t) \cdot c)^2} \tag{1.74}$$

By substituting the expression for momentum $p(t)$ into the total energy equation, we obtain:

$$E(t) = \sqrt{(m_0c^2)^2 + (e \cdot B(t) \cdot \rho \cdot c)^2} \tag{1.75}$$

Substituting the expression for velocity $v(t)$ into the equation for the revolution frequency, we have:

$$f_{\text{rev}}(t) = \frac{1}{2\pi R_s} \cdot \frac{e \cdot B(t) \cdot \rho \cdot c^2}{\sqrt{(m_0c^2)^2 + (e \cdot B(t) \cdot \rho \cdot c)^2}} \tag{1.76}$$

Finally, this leads to the expression for the ratio of the RF to the harmonic number, showing how it depends on the magnetic field $B(t)$, the particle's rest mass m_0, the bending radius ρ, and the synchrotron radius R_s:

$$\frac{f_{\text{RF}}(t)}{h} = \frac{c}{2\pi R_s} \cdot \frac{B(t)}{\sqrt{B(t)^2 + \left(\frac{m_0 \cdot c}{e \cdot \rho}\right)^2}} \tag{1.77}$$

The above equation shows how the frequency of the RF system in a synchrotron must be adjusted to stay in synchronization with the particles as they move around the ring. It takes into account the strength of the magnetic field, the mass of the particles, and how fast they are moving. As the particles gain energy and speed up, the magnetic field changes, and this equation helps ensure that the RF system keeps accelerating the particles efficiently by staying perfectly timed with their movement.

1.6 Synchrocyclotron

To overcome the energy limitations of conventional cyclotrons, frequency-modulated cyclotrons, known as synchrocyclotrons, were developed. It is a type of particle accelerator that combines the principles of both a cyclotron and a synchrotron. It operates similarly to a cyclotron but incorporates a technique called *synchrotron resonance* to account for relativistic effects as particles approach the speed of light. As the particles gain energy and their velocity increases, their revolution frequency decreases due to relativistic mass increase. To maintain synchronization, the synchrocyclotron adjusts the frequency of the applied RF electric field to match the changing revolution frequency of the particles. This adjustment allows particles to continue gaining energy efficiently at relativistic speeds. These accelerators utilize the principle of phase stability discovered by Veksler [11], which enables particles to remain in phase with the RF field during acceleration. By gradually lowering the RF to match the decreasing revolution frequency of the accelerating protons, synchrocyclotrons have successfully accelerated protons to energies of several hundred MeV. In synchrocyclotrons, the magnetic field decreases radially to provide axial focusing, ensuring the beam remains stable. However, since the frequency must be continuously reduced during acceleration, synchrocyclotrons produce pulsed beams with a low duty factor and relatively low average beam intensities compared to other accelerators.

The magnetic field B, particle charge q, and mass m determine the motion of the particle in the synchrocyclotron. As the particle accelerates, its mass increases due to relativistic effects, which is represented by the relativistic mass formula:

$$m' = \frac{m}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \tag{1.78}$$

where v is the particle's velocity and c is the speed of light.

The cyclotron frequency f decreases as the particle reaches relativistic speeds and is given by:

$$f = \frac{qB}{2\pi m'} = \frac{qB}{2\pi} \cdot \frac{\sqrt{1-\left(\frac{v}{c}\right)^2}}{m} \tag{1.79}$$

For nonrelativistic particles, the frequency f_0 remains constant and is expressed as:

$$f_0 = \frac{qB}{2\pi m} \tag{1.80}$$

The relationship between the relativistic frequency f and the nonrelativistic frequency f_0 is:

$$\frac{f}{f_0} = \sqrt{1-\left(\frac{v}{c}\right)^2} \tag{1.81}$$

This allows us to find the velocity v as:

$$\frac{v}{c} = \sqrt{1-\left(\frac{f}{f_0}\right)^2} \tag{1.82}$$

The velocity v is also related to the magnetic field B and the radius R of the particle's path:

$$v = \frac{qBR}{m} \tag{1.83}$$

Substituting this into the previous expression gives:

$$\frac{qBR}{mc} = \sqrt{1-\left(\frac{f}{f_0}\right)^2} \tag{1.84}$$

Solving for B gives the expression:

$$B = \frac{mc}{qR}\sqrt{\left(\frac{f}{f_0}\right)^2 - 1} \tag{1.85}$$

This equation shows how the magnetic field adjusts due to the relativistic effects as the particle accelerates. The resonance condition is achieved when the particle's revolution frequency f_0 synchronizes with the frequency f of the RF field, maintaining efficient acceleration:

$$f_0 = nf \tag{1.86}$$

where n is an integer representing the harmonic number.

The synchrocyclotron represents a significant advancement in particle accelerator technology by overcoming the limitations of earlier cyclotrons, particularly with regard to relativistic effects. Through the synchronization of the RF field with the changing frequency of the particles' revolution, synchrocyclotrons efficiently accelerate particles to near-light speeds. This innovation has paved the way for numerous applications, including advancements in medical radiation therapy and nuclear physics research. As particle velocities approach the speed of light, the need to account for relativistic mass increases becomes critical. The formulas governing the relationship between magnetic fields, particle velocity, and energy ensure stable acceleration while maintaining resonance conditions. The insights provided by the relativistic effects on frequency and mass help optimize the design of synchrocyclotrons for high-energy applications.

1.7 Betatron

The betatron, an electron accelerator, was developed in the 1930s by Donald W. Kerst [12] at the University of Illinois, USA. It is designed to accelerate charged particles using rapidly varying magnetic fields, which induce an electric field that accelerates the particles. The particles follow a spiral path within a toroidal, or doughnut-shaped, vacuum chamber. Betatrons are nonrelativistic accelerators and typically operate in the RF range. While betatrons are not as prominent as LINACs today, they were significant in the early development of particle physics, as well as in medical and industrial applications. Betatrons operate on the principle of electromagnetic induction:

$$-e\oint \vec{E}.\overrightarrow{dl} = -\frac{d}{dt}\int \vec{B}.\overrightarrow{dA} \tag{1.87}$$

The induced electromotive force (emf) is related to the rate of change of magnetic flux Φ, where Φ is the flux through the area enclosed by the orbit:

$$\mathcal{E} = -\frac{d\Phi}{dt}, \quad \Phi = \pi r^2\langle B\rangle, \quad \frac{d\Phi}{dt} = \pi r^2\frac{d\langle B\rangle}{dt} \tag{1.88}$$

where $\mathcal{E}$ is the induced emf, Φ is the magnetic flux through the orbit, r is the radius of the orbit, $\langle B\rangle$ is the average magnetic field over the area enclosed by the orbit, and $\frac{d\Phi}{dt}$ is the rate of change of the magnetic flux.

The induced emf $\mathcal{E}$ around the orbit is related to the induced electric field E, which depends on the rate of change of the average magnetic field:

$$\mathcal{E} = E\cdot 2\pi r, \quad E = \frac{r}{2}\cdot\frac{d\langle B\rangle}{dt} \tag{1.89}$$

where E is the induced electric field around the orbit, and $2\pi r$ is the circumference of the orbit.

The energy gained by a particle per revolution is proportional to the induced electric field around the orbit and can be expressed as:

$$\Delta\mathcal{E} = -e\cdot\mathcal{E} = -e\cdot 2\pi rE = -e\cdot\pi r^2\frac{d\langle B\rangle}{dt} \tag{1.90}$$

This equation shows how the energy gained by the particle in each revolution is tied to the rate of change of the magnetic flux inside the orbit.

The Lorentz force provides the centripetal force necessary to keep the particle in a circular orbit, balanced by the work done by the magnetic field:

$$qvB_{\text{orbit}} = \frac{mv^2}{r}, \quad qvB_{\text{orbit}}r = q\pi r^2 \frac{d\langle B\rangle}{dt} \tag{1.91}$$

where q is the charge of the particle, v is the velocity of the particle, B_{orbit} is the magnetic field at the orbit, m is the mass of the particle, and $\frac{mv^2}{r}$ is the centripetal force acting on the particle.

The **betatron frequency**, or the frequency at which the particles revolve in the circular path due to the magnetic field, is given by:

$$\omega = \frac{qB_{\text{orbit}}}{m} \tag{1.92}$$

where ω is the angular frequency, q is the charge of the particle, and B_{orbit} is the magnetic field at the particle's orbit.

This condition ensures the magnetic field at the orbit is half the average magnetic field within the orbit, allowing stable motion in the betatron:

$$B_{\text{orbit}} = \frac{1}{2}\langle B\rangle \tag{1.93}$$

This condition is crucial for ensuring stable orbits in a betatron. It signifies that the magnetic field at the radius of the particle's orbit must be half the average magnetic field within the area enclosed by the orbit. This balance is necessary to provide the correct centripetal force to keep the particles moving in a stable circular path while they are being accelerated by the induced electric field. If this condition is not met, the particles will either spiral inward or outward, leading to instability. By maintaining this relationship, the betatron can efficiently accelerate particles to high energies while ensuring they remain in a stable orbit.

1.8 Particle Colliders

Particle colliders, as the name suggests, accelerate particles and then collide them to break them apart into more fundamental entities, enabling the exploration of the subatomic world. Particle colliders are specialized types of accelerators used in experimental physics to study the fundamental properties of particles and the forces of nature. Colliders have been instrumental in discovering particles such as the Higgs boson and the top quark, among others. Colliders are designed to achieve high collision rates per unit of time, known as high luminosity, which is essential for increasing the probability of rare particle interactions. Colliders can be categorized into two types:

1. **Circular colliders:** These accelerators guide particles in a circular path using magnetic fields, with collisions occurring at experimental stations along the circumference. Examples include the LHC at CERN, Geneva, Switzerland, and the Relativistic Heavy Ion Collider at Brookhaven National Laboratory, USA.

2. **Linear colliders (LINACs):** These accelerators propel particles in a straight line using RF cavities. Examples include the Stanford Linear Collider and the future International Linear Collider.

The collision energy in a collider is a critical parameter for experiments and is dependent on the energy (E) of the colliding particle. The total energy and momentum of the colliding particles are important parameters for determining the outcome of particle collisions. In symmetric colliders, such as the LHC, the particles in both beams have equal and opposite momentum, leading to maximum center-of-mass energy, given by:

$$E_{\text{collision}} = 2E \tag{1.94}$$

The luminosity of a collider is given by:

$$L = \frac{N_1 N_2 f}{4\pi\sigma_x\sigma_y} \tag{1.95}$$

where N_1 and N_2 are the number of particles per bunch for each beam, f is the bunch collision frequency, σ_x and σ_y are the transverse beam sizes in the horizontal and vertical directions, respectively.

1.8.1 Microtron

A microtron [13, 14] is a type of circular particle accelerator that utilizes a combination of magnetic and electric fields to accelerate charged particles, typically electrons, to high energies. It consists of several key components, including a uniform magnetic field to steer particles in a circular orbit, RF cavities to accelerate particles, and an injection system to introduce particles into the accelerator. The operation of a microtron is based on the principle of phase stability and the synchronization between the particle motion and the applied RF field.

The working of a microtron begins with the injection of charged particles, usually electrons, into the accelerator with an initial kinetic energy K_0. Once inside, the particles are guided in a circular orbit by a uniform magnetic field B, which exerts a Lorentz force on them. The radius of the circular orbit is given by the expression

$$r = \frac{\gamma m v}{qB}$$

where γ is the relativistic factor $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, m is the rest mass of the electron, v is its velocity, q is the charge of the particle, and B is the applied magnetic field.

As the particles traverse their circular paths, they pass through an RF cavity at regular intervals, gaining an increment of energy ΔK each time. The energy gain per pass can be expressed as:

$$\Delta K = qV_{\text{RF}} \cos\phi$$

where V_{RF} is the peak RF voltage and ϕ is the phase of the particle relative to the RF field. To ensure efficient acceleration, the phase stability condition must be maintained, allowing the particles to synchronize with the RF field and arrive at the accelerating gaps at the correct time.

A unique feature of the microtron is that, despite the continuous increase in particle energy, the revolution period remains constant. This is because the increase in energy leads to a proportional increase in orbit radius, ensuring that the total time taken per revolution, given by

$$T = \frac{2\pi r_n}{v}$$

remains unchanged. This characteristic simplifies the synchronization process and distinguishes the microtron from other accelerators.

Another important aspect of microtron operation is the gradual expansion of the particle orbit. With each successive acceleration cycle, the orbit radius increases stepwise according to the relation

$$r_n = r_0 + n\Delta r$$

where r_0 is the initial radius and n represents the number of revolutions. The uniform magnetic field ensures constant transverse focusing, which helps maintain beam stability and prevents excessive divergence.

Microtrons are widely used due to their compact design, ease of synchronization, and ability to achieve high energies with relatively low operational costs. These accelerators have found significant applications in the medical field, particularly in radiation therapy for cancer treatment, where electron beams are used to precisely target tumors while minimizing damage to surrounding tissues. In materials science, microtrons are employed to study material properties through electron beam interactions, providing valuable insights into structural composition. Industrial applications also benefit from microtrons, such as in the sterilization of medical equipment, food irradiation for preservation, and polymer modification processes.

In conclusion, the microtron represents an efficient and versatile particle accelerator, capable of providing high-energy electron beams for a range of applications. Its unique properties, such as constant revolution time and phase stability, make it a valuable tool in both scientific research and industrial settings.

1.9 FFAG Accelerators

The acronym FFAG stands for “fixed-field alternating gradient.” As the name suggests, these use AG focusing, but with fixed magnetic fields. FFAG accelerators [15] were first conceived in the early 1950s as a method to reduce the accelerator radius compared to traditional circular accelerators. Despite their promising potential, the development of FFAGs faced delays primarily due to the complexity of their magnet designs, which were more intricate compared to those of the rapidly advancing synchrotron technology. Among the various designs, the “radial-sector FFAG” represents the most straightforward implementation. In FFAG accelerators, AG focusing is applied with a series of dipole magnets. This approach replaces a continuous magnetic field with discrete focusing and defocusing elements. This configuration not only provides the necessary beam focusing but also creates gaps to accommodate accelerating cavities and other essential components for beam injection, extraction, and monitoring. FFAG accelerators combine features of synchrotrons and cyclotrons, using a fixed magnetic field and alternating focusing to accelerate particles continuously across a wide range of energies.

FFAG accelerators [16] combine features of both LINACs and circular accelerators. FFAG accelerators use a fixed magnetic field strength with alternating magnetic field gradients to accelerate charged particles. Unlike synchrotron accelerators, where the magnetic field strength increases along the path of the particles, in FFAG accelerators, the magnetic field remains constant while the gradient of the magnetic field changes over time. Particles follow a cycloidal motion within the magnetic field. The equation of motion for a particle's trajectory in an FFAG accelerator can be complex, involving a combination of trigonometric functions and elliptical integrals. More detailed study can be found in Refs. [16–18].

There are two types of FFAGs: scaling and nonscaling. In a scaling FFAG, the focusing of particles remains constant regardless of their energy. The magnetic field must be carefully designed to ensure that particles follow similar orbits at all energies. This design is beneficial for maintaining stable particle paths over a wide energy range, but the strict requirements for the magnetic field make it more challenging to construct.

In contrast, nonscaling FFAGs [18] allow the focusing to change as particle energy increases, meaning that particle orbits can vary. This design offers greater flexibility, as particles can follow different paths at different energies, which is known as chromatic behavior. The magnetic field in a nonscaling FFAG is easier to design, making the overall system simpler and more compact. However, this flexibility can lead to less stability in particle orbits, especially at higher energies, as the precise control over the paths is reduced.

Scaling and nonscaling FFAG accelerators have different designs and characteristics. Scaling FFAG accelerators are designed with magnetic fields that change with energy in a specific way. The magnetic field strength $B(r)$ in a scaling FFAG varies with the radius r such that $B(r) \propto \frac{1}{r}$. This design helps maintain consistent transverse focusing across different energy levels, which leads to stable and predictable beam dynamics over a wide range of energies. However, this approach requires precise control and complex design to ensure that focusing remains uniform.

In contrast, nonscaling FFAG accelerators use magnetic fields that do not necessarily follow this scaling rule. The field profile can vary in a more complex manner, and the transverse focusing also changes with radius and energy. This can result in more complicated beam dynamics. Nonscaling FFAGs offer greater flexibility in energy range and field design but do not provide the same level of predictable stability as scaling FFAGs.

A comparison of various circular accelerators is given in Table 1.2.

Table 1.2 Comparison of circular accelerators.

Accelerator type	Revolution time	Orbit radius	Transverse focusing
Cyclotron	Constant	Variable	Variable
Synchrotron	Variable (except relativistic)	Constant	Constant
Nonscaling FFAG	Variable (small)	Variable (small)	Variable
Scaling FFAG	Variable	Variable	Constant
Betatron	Variable	Constant	Variable
Microtron	Constant	Variable	Constant

Curiosity: *Have you ever watched a swimmer ride a big ocean wave all the way to the shore? Now imagine a tiny particle doing the same thing—not in water, but on a wave made of electric fields inside a special gas called plasma. This is how wakefield acceleration works. When a strong laser pulse or a fast-moving particle beam travels through plasma, it pushes away the electrons and creates a wave behind it—just like a boat makes waves in water. If we place other particles at just the right spot on this wave, they can "surf" it and get pushed forward really fast, gaining a lot of energy in a short distance. Could this help us build smaller and cheaper accelerators that don't need kilometers of space? Why is plasma so good at making strong waves, and how does it help us go beyond what normal machines can do? What happens if a particle ends up in the wrong part of the wave—does it slow down instead of speeding up? How do laser-driven and particle-driven wakefields make these waves differently? And could we one day use this technology for new kinds of medical tools or even space engines? We try to answer some of these exciting questions here, but many more are still waiting for curious minds to explore!*

1.10 Wakefield Accelerators

Conventional accelerators are large and expensive, with maximum achievable fields limited to a few tens of MV/m due to breakdown issues. Plasma waves offer an alternative, enabling more compact and powerful acceleration. Analogous to a boat creating a wake in water, a laser pulse traveling at the speed of light through a plasma generates a wake behind it, termed an electron plasma oscillation. These high-frequency oscillations of electrons in the plasma result in strong electric fields over short distances, reaching the GV/m range. The wakefield amplitude is maximized when the drive duration is shorter than the plasma wavelength.

1.10.1 Laser Wakefield Acceleration

A laser plasma wakefield accelerator (LWFA) [19] uses a powerful, very short laser pulse to push electrons in a plasma, creating wakes. These wakes have strong electric fields that can accelerate charged particles to very high energies in a small space. It is much smaller than traditional particle accelerators. The wakefield can be linear or nonlinear, depending on whether the driver beam density is less than or greater than the plasma electron density, respectively. At high driver densities, a blowout regime occurs, allowing for linear focusing and higher acceleration efficiency.

In LWFA, several fundamental physical quantities are essential to describe the behavior of electrons in intense electromagnetic fields. The normalized vector potential, denoted by a_0, represents the strength of the laser field and determines whether the electron motion becomes relativistic. The resulting electron oscillations in the laser field are described by the quiver velocity, v_q, which increases with a_0. The laser field itself is characterized by its peak electric field E_0, which is directly related to the laser intensity I_0. The laser also has a specific angular frequency ω_0 and wavelength λ_0, which define its temporal and spatial scales. While a free electron in a vacuum plane wave cannot gain net energy due to the Lawson–Woodward theorem, in a plasma, the presence of spatial inhomogeneities allows for net energy gain. The electron experiences a net outward push due to the ponderomotive force, which arises from gradients in the electric field amplitude. The plasma's response is governed by the plasma frequency ω_p, which depends on the electron density n_e, and the associated plasma wavelength λ_p sets the characteristic scale of the wakefield. The longitudinal accelerating field E_p, generated behind the laser pulse, is responsible for electron acceleration. However, electrons eventually outrun the accelerating phase, which defines the dephasing length, denoted as L_d. These quantities together form the foundation for understanding energy transfer from laser pulses to particles in plasma.

The normalized vector potential a_0 describes the behavior of an electron in an intense laser field and is defined as:

$$a_0 = \frac{eE_0}{m_e \omega_0 c}$$

The quiver velocity v_q of the electron in the laser field is:

$$v_q \approx c \frac{a_0}{\sqrt{1 + a_0^2}}$$

In the nonrelativistic regime ($a_0 \ll 1$), the quiver velocity v_q is:

$$v_q = \frac{eE_0}{m_e \omega_0}$$

In the relativistic regime ($a_0 \gtrsim 1$), the quiver velocity approaches the speed of light. The peak electric field E_0 is related to the laser intensity I_0 by:

$$E_0 = \sqrt{\frac{2I_0}{\epsilon_0 c}}$$

Substituting E_0 and $\omega_0 = \frac{2\pi c}{\lambda_0}$ into the expression for a_0, we get:

$$a_0 = \frac{e\lambda_0}{2\pi m_e c^2} \times \sqrt{\frac{2I_0}{\epsilon_0}}$$

For a laser with wavelength $\lambda_0 = 1\,\mu\text{m}$, a_0 is related to the intensity I_0 by:

$$a_0 = 0.855 \times 10^{-9} \times \sqrt{I_0[\text{W/cm}^2]}$$

The Lawson–Woodward theorem states that a free electron cannot gain net energy from a plane electromagnetic wave in vacuum. The force on the electron is described by the Lorentz equation:

$$\frac{d\mathbf{p}}{dt} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

The rate of energy transfer is:

$$\frac{d\mathcal{E}}{dt} = q\mathbf{v} \cdot \mathbf{E}$$

In the case of a plane wave, the work done by the field averages out to zero over a full cycle:

$$\Delta\mathcal{E} = q \int_0^T \mathbf{v}(t) \cdot \mathbf{E}(t)\, dt = 0$$

Thus, no net energy is transferred to the particle. Schemes like laser wakefield acceleration (LWFA) introduce plasma to break the symmetry and enable efficient particle acceleration.

The motion of an electron in an oscillating electric field $\mathbf{E}(t) = \mathbf{E}_s(\mathbf{r})\cos(\omega t)$ is described by the Lorentz force:

$$m_e \frac{d^2\mathbf{r}}{dt^2} = e\mathbf{E}_s(\mathbf{r})\cos(\omega t)$$

The velocity $\mathbf{v}(t)$ is:

$$\mathbf{v}(t) = \frac{e\mathbf{E}_s(\mathbf{r})}{m_e\omega}\sin(\omega t)$$

The time-averaged kinetic energy of the electron is:

$$\langle \mathcal{E}_k \rangle = \frac{e^2 E_s^2(\mathbf{r})}{4m_e\omega^2}$$

The ponderomotive force is the gradient of the time-averaged kinetic energy:

$$\mathbf{F}_\text{p} = -\frac{e^2}{4m_e\omega^2}\nabla E_s^2(\mathbf{r})$$

The plasma frequency ω_p is defined as:

$$\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$$

The plasma wavelength λ_p is related to ω_p by:

$$\lambda_p = \frac{2\pi c}{\omega_p}$$

Simplifying:

$$\lambda_p[\mu\text{m}] = 33 \times \left(n_e(10^{18}\,\text{cm}^{-3})\right)^{-1/2}$$

The ponderomotive force creates wakefields that generate an accelerating electric field E_p, proportional to ω_p and the density perturbation $\frac{\delta n_e}{n_e}$. The strength of the field is:

$$E_p[\text{GV/m}] \approx 96 \times \left(n_e(10^{18}\ \text{cm}^{-3})\right)^{1/2} \frac{\delta n_e}{n_e}$$

The laser frequency and wavelength are related similarly. We then express the frequency ratio in terms of wavelengths:

$$\omega_0 = \frac{2\pi c}{\lambda_0}, \qquad \left(\frac{\omega_p}{\omega_0}\right)^2 = \left(\frac{\lambda_0}{\lambda_p}\right)^2 \tag{1.96}$$

In an underdense plasma, the phase velocity of the plasma wave is close to the speed of light. It is approximately given by the laser's group velocity:

$$\beta_p = \frac{v_p}{c} \approx \sqrt{1 - \left(\frac{\omega_p}{\omega_0}\right)^2}, \qquad \text{(from cold plasma dispersion)} \tag{1.97}$$

Applying a binomial expansion for small ω_p/ω_0, we obtain:

$$\beta_p \approx 1 - \frac{1}{2}\left(\frac{\omega_p}{\omega_0}\right)^2, \ \text{ or in terms of wavelengths: } \beta_p \approx 1 - \frac{1}{2}\left(\frac{\lambda_0}{\lambda_p}\right)^2 \tag{1.98}$$

The dephasing length is the distance over which a relativistic electron (moving at nearly c) slips forward by one accelerating half-period of the plasma wave:

$$L_d = \frac{\lambda_p}{2(1 - \beta_p)} \tag{1.99}$$

Substituting the approximation for β_p into the expression:

$$L_d = \frac{\lambda_p}{2 \cdot \frac{1}{2}\left(\frac{\lambda_0}{\lambda_p}\right)^2} = \frac{\lambda_p}{\left(\frac{\lambda_0}{\lambda_p}\right)^2} = \frac{\lambda_p^3}{\lambda_0^2} \tag{1.100}$$

This gives the final 1D expression for dephasing length:

$$\boxed{L_d = \frac{\lambda_p^3}{\lambda_0^2}} \tag{1.101}$$

To rewrite in terms of frequencies, recall:

$$\lambda_p = \frac{2\pi c}{\omega_p}, \quad \lambda_0 = \frac{2\pi c}{\omega_0} \tag{1.102}$$

Substituting these into the wavelength form of L_d, we get:

$$L_d = \frac{\left(\frac{2\pi c}{\omega_p}\right)^3}{\left(\frac{2\pi c}{\omega_0}\right)^2} = \frac{2\pi c \cdot \omega_0^2}{\omega_p^3} \tag{1.103}$$

This gives the dephasing length in terms of plasma and laser frequency:

$$\boxed{L_d = \frac{2\pi c\,\omega_0^2}{\omega_p^3}} \tag{1.104}$$

In higher dimensions, the expression for dephasing length changes due to transverse effects and nonlinear wake evolution.

In the 2D nonlinear regime:

$$L_d^{2D} = \frac{\omega_0^2}{2\omega_p^2}\lambda_p \tag{1.105}$$

In the 3D bubble regime, where the wake forms a near-spherical cavity, the dephasing length becomes:

$$L_d^{3D} = \frac{4}{3} \cdot \frac{\omega_0^2}{\omega_p^2} \cdot \frac{\sqrt{a_0}}{k_p}, \qquad \text{with } k_p = \frac{\omega_p}{c} \tag{1.106}$$

These expressions show how dephasing length depends strongly on the plasma density and laser wavelength, setting a fundamental limit on how far an electron can be accelerated in a single LWFA stage.

1.10.2 Beam Wakefield Acceleration

In *beam-driven plasma wakefield acceleration* (PWFA) [20], a high-energy particle beam, called the *drive beam*, travels through a plasma and excites oscillations in the plasma electrons. These oscillations create a wakefield, which can accelerate another particle beam, called the *witness beam*. The witness beam is placed behind the drive beam to gain energy from the wakefield.

The plasma density, denoted by n_e, is the number of electrons per cubic centimeter. The plasma frequency ω_p, which describes the natural oscillation frequency of the plasma electrons, depends on n_e, the electron charge e, electron mass m_e, and the permittivity of free space ϵ_0. The wake created by the drive beam has a characteristic wavelength, called the *plasma wavelength*, denoted λ_p, which represents the distance between consecutive peaks in the plasma oscillations.

The strength of the electric field created by the wake, represented as E_p, accelerates the witness beam. The wakefield potential, denoted Φ, describes the potential field responsible for generating the wake, and it is directly linked to the charge density of the drive beam, n_b.

The plasma frequency ω_p is given by:

$$\omega_p = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}}$$

This formula shows that the plasma frequency depends on the plasma density. A higher electron density leads to a higher plasma frequency.

The plasma wavelength λ_p, which is the distance between the peaks of the oscillations, is given by:

$$\lambda_p = \frac{2\pi c}{\omega_p}$$

Here, c is the speed of light, and λ_p is inversely proportional to the plasma frequency. The wakefield potential Φ excited by the drive beam is given by:

$$\Phi \approx \frac{en_b}{\epsilon_0 k_p^2}$$

Here, $k_p = \frac{\omega_p}{c}$ is the plasma wavenumber. The longitudinal electric field in the plasma wake, denoted as E_z, is responsible for accelerating particles, and it is proportional to the plasma frequency ω_p. It can be expressed as:

$$E_z \approx \frac{m_e c \omega_p}{e}$$

The energy gained by the witness beam as it travels through the plasma is given by:

$$\Delta\mathcal{E} = m_e c^2 \frac{L}{\lambda_p}$$

Here, L is the length of the plasma, and λ_p is the plasma wavelength. Finally, a practical formula that estimates the peak electric field in beam-driven PWFA is given by:

$$E_p[\text{GV/m}] \approx 96 \times \left(n_e(10^{18}\ \text{cm}^{-3})\right)^{1/2} \frac{\delta n_e}{n_e}$$

This formula shows that the peak electric field E_p scales with the square root of the plasma density n_e and is proportional to the relative plasma density perturbation $\frac{\delta n_e}{n_e}$. These formulae provide an understanding of how beam-driven wakefield acceleration works and how the electric field and energy gain scale with plasma density.

Wakefield accelerators are an exciting new development in the quest for smaller, high-energy particle accelerators. By using plasma waves, these accelerators can potentially overcome the limitations of traditional accelerators, reaching electric fields in the range of GV/m. Both LWFA and PWFA offer unique benefits, including the ability to create very strong acceleration fields over short distances, making them ideal for the next generation of accelerators used in scientific and medical applications. Key concepts such as the ponderomotive force, plasma frequency, and dephasing length are essential to understanding how particle acceleration works in these systems. In advanced regimes, like the bubble regime, wakefield accelerators can achieve higher energy transfer efficiency, which is important for developing practical and scalable devices. Despite their potential, there are still challenges to be solved, such as improving beam quality, managing plasma density, and ensuring efficient energy transfer. As research continues and technology improves, these issues are expected to be resolved, moving wakefield accelerators closer to widespread use. These accelerators have the potential to transform particle physics by providing more accessible and affordable accelerators for a variety of purposes.

Curiosity: *In the late 1890s, a series of amazing discoveries changed science forever. In 1895, Wilhelm Roentgen accidentally found X-rays—an invisible kind of energy that could pass through things like skin and show bones. This inspired Henri Becquerel, who in 1896 discovered that uranium gave off a similar invisible energy all by itself, even in the dark.*

Soon after, Marie and Pierre Curie studied these rays and discovered two new elements—polonium and radium—and called this kind of energy "radioactivity." They all worked hard and made great sacrifices, and between 1901 and 1903, Roentgen, Becquerel, and the Curies won Nobel Prizes. Their discoveries helped start the fields of nuclear science, medical X-rays, and cancer treatment using radiation. Have you ever wondered what radiation really is, and why we take such care to protect ourselves from it in particle accelerators? Radiation isn't always dangerous—it's simply energy that travels through space as particles or waves. Some types, like light and radio waves, are harmless. But when particles move very fast, like in an accelerator, they can create stronger kinds of radiation, like X-rays or neutrons, which can pass through materials and damage living cells. That's why we build thick walls around accelerator areas and use special shielding to block harmful radiation from escaping. But what makes some materials radioactive in the first place? It turns out that certain atoms are naturally unstable—they slowly break apart and give off energy in the form of radiation. This process, called radioactivity, happens in nature (like in rocks and the Sun) but can also happen when accelerator beams hit a target and make new, unstable atoms. How do we measure radiation? And how much is safe? People working near accelerators wear special badges that record how much radiation they're exposed to, and safety rules help limit their dose. Some of the particles created in accelerators disappear quickly, while others stay radioactive for a long time—so how do we safely store or dispose of them? These are just a few of the big questions in radiation physics and protection. We've touched on some of them here, but there's a lot more to discover about how radiation behaves, how it's controlled, and how we stay safe while using it for science and medicine.

1.11 Radiation Physics

Several scientists made major contributions to radiation physics, and their names are used in key units today. Henri Becquerel found radioactivity in 1896, which led to the Becquerel (Bq), a unit measuring the number of decays per second. Marie and Pierre Curie, known for discovering radium and polonium, are honored with the Curie (Ci), used for larger amounts of radioactivity. Louis Harold Gray studied how radiation affects tissues, resulting in the Gray (Gy), a unit for absorbed dose. Rolf Sievert worked on radiation's biological effects, leading to the Sievert (Sv), which considers the type of radiation. Wilhelm Conrad Röntgen, who discovered X-rays, is remembered through the Roentgen (R), which measures radiation exposure in air. These scientists laid the foundation for understanding and measuring radiation. Radiation physics plays a vital role in particle accelerators, where ensuring the safety of radiation workers, especially accelerator staff, is of utmost importance. In this context, radiation refers to both direct and secondary radiation produced in accelerators. The level of radioactivity is commonly measured in Becquerels (Bq), which indicates the number of nuclear decays per second. Specifically, 1 Bq means one decay per second. A higher number of Bq signifies that the substance is undergoing more nuclear decays in a given time, indicating greater radioactivity. However, a higher level of radioactivity does not necessarily equate to higher danger. The type of radiation (such as alpha, beta, or gamma), its energy, and how it interacts with matter are crucial factors that determine its biological effects. For large amounts of radioactivity, the Ci is often used because it represents a much higher level of radioactive decay. **1 Ci =** 3.7×10^{10} **Bq**, making the Ci more convenient for describing high levels of radioactivity.

In radiation physics, different units are used to measure various aspects of radiation, especially in accelerator environments. The following is a list of the common units in radiation physics:

- Absorbed dose (Gray [Gy]): The Gy is the SI unit for absorbed dose, which represents the amount of energy absorbed from radiation per unit mass. One Gray is one joule per kilogram of tissue.
  1 Gy = 100 Rad.
- Absorbed dose (Rad): The Rad (radiation absorbed dose) is an older unit of absorbed dose.
  1 Rad = 0.01 Gy.
- Equivalent dose (Sievert [Sv]): The Sv is the SI unit for equivalent dose, which accounts for the biological impact of radiation. It is calculated by multiplying the absorbed dose (in Gy) by a radiation weighting factor based on the type of radiation.
  1 Sv = 100 Rem.
- Equivalent dose (Rem): The Rem (Roentgen equivalent man) is an older unit of equivalent dose.
  1 Rem = 0.01 Sv.
- Exposure (Roentgen [R]): The R is used to measure radiation exposure in air, particularly for X-rays and gamma rays. **1 R is** 2.58×10^{-4} **Coulombs per kilogram of air.** For X-rays and gamma rays in air, **1 R ≈ 0.009 Gy and 0.009 Sv.**

Let A be the activity of the radioactive source, measured in curies (Ci) or becquerels (Bq). The distance from the source is represented by d in meters. The gamma constant, Γ, is specific to the type of radiation and is measured in $\text{mSv} \cdot \text{m}^2/\text{Ci} \cdot \text{h}$. The dose rate, $\dot{D}$, is the radiation intensity at a given distance, expressed in millisieverts per hour (mSv/h) or rem per hour (rem/h). If shielding is involved, I_0 represents the initial dose rate before the shield, and the shield's thickness is given by x in centimeters. The linear attenuation coefficient μ, which is specific to the shielding material and radiation energy, is measured in cm^{-1}. The reduced dose rate after shielding is H. The total dose received over an exposure time t, measured in hours, is denoted as D. Finally, radioactive decay is described by a decay constant λ, which relates to how fast a substance decays over time, with t representing the elapsed time. The half-value layer (HVL) gives the thickness of the shielding material needed to reduce the radiation intensity by half.

The dose rate at a certain distance d from a radiation source is calculated using the inverse square law:

$$\dot{D} = \frac{\Gamma \cdot A}{d^2} \tag{1.107}$$

When radiation passes through a shield of thickness x, the dose rate after shielding is given as follows:

$$I = I_0 \cdot e^{-\mu x} \tag{1.108}$$

The total dose received over a time t, based on the dose rate after shielding, is given as follows:

$$D = H \cdot t \tag{1.109}$$

The radioactive decay formula, showing how the activity $A(t)$ changes over time, is given as follows:

$$A(t) = A_0 \cdot e^{-\lambda t} \tag{1.110}$$

The formula for the HVL, which gives the thickness required to reduce the radiation intensity by half, is given as follows:

$$HVL = \frac{\ln(2)}{\mu} \tag{1.111}$$

The total radiation exposure, or dose limit, can be calculated is given as follows:

$$D = H \cdot t \tag{1.112}$$

As per exponential radioactive decay law, Let N_0 and N be the number of radioactive atoms of a sample at $t = 0$ and time (t)

$$N = N_0 \exp(-\lambda t)$$

The activity of a radioactive source quantifies the rate at which atom in a radioactive source decays. It is number of disintegration per unit time. Activity of sample also follows same rule:

$$A = A_0 \exp(-\lambda t)$$

Half life of a radionuclide is defined as time in which activity of a sample reduce to half. Thus, One can write:

$$\frac{1}{2} = \exp(-\lambda T_{1/2})$$

$$A = \lambda N = \frac{N \ln 2}{T_{1/2}}$$

where $T_{1/2}$ is the half life of the source.

Next, the specific activity of sample is defined as its activity per unit mass.

$$SA = \frac{N_A \lambda}{M}$$

where NA is the Avogadro's number, M is the atomic weight.

In practical units:

$$SA(Bq/g) = \frac{4.17 \times 10^{23}}{M(amu) T_{1/2}(sec)}$$

Radiation physics is essential for understanding and controlling radiation, especially in places like particle accelerators, where safety is a major concern. The contributions of pioneering scientists such as Henri Becquerel, Marie and Pierre Curie, Louis Harold Gray, Rolf Sievert, and Wilhelm Conrad Röntgen have provided the foundation for modern radiation measurement systems. We now have key units such as the Becquerel, Curie, Gray, Sievert, and Roentgen to measure different types of radiation and its effects.

Curiosity: *Have you ever thought about how hard it is to keep a group of like-charged particles together? Since they all repel each other, they never really want to stay close. But in particle accelerators, we have to guide these fast-moving particles from their source all the way to a target. How do we do that? We use electric and magnetic fields to steer and control the beam, like invisible hands guiding it along a path. Along the way, we often need to check what kind of particles are in the beam. By passing them through special fields, we can bend them in different ways and learn about their speed and type. To keep the beam sharp and narrow, we bunch the particles closer together in the direction they're moving and also focus them from the sides. This helps control the size and shape of the beam. But first, we need to pull the particles out of the source, and we do that by giving them energy using high voltage. Once the beam is moving through the vacuum inside the accelerator, something interesting happens—if we don't keep focusing it, the beam slowly spreads out in both directions. This shows how carefully we have to control everything to keep the beam stable and useful. We've looked at some of these key ideas here, but there's still much more to explore about how we guide and shape beams of particles with precision.*

1.12 Beam Conceptual Visualization

Finally, from Figure 1.6, one may get an idea of various effects over beams in the simple environment of electromagnetic fields and some basic properties of beams.

a. **Beam Collection of Charges (Coulomb Forces)** Charged particles in the beam interact through Coulomb forces, causing space charge effects that lead to beam expansion.
b. **Beam Transport from Ion Source to Target** Beam transport involves guiding and aligning the beam using electrostatic lenses, magnetic quadrupoles, and drift tubes. This process ensures minimal emittance growth and maintains beam quality.
c. **Beam Bending via Electric or Magnetic Dipole** Electric dipoles deflect beams using electric fields, while magnetic dipoles use the Lorentz force for precise bending. This technique is commonly used in particle accelerators.
d. **Beam Bunching and Focusing** Beam bunching compresses continuous streams into periodic bunches using RF cavities. Focusing counters divergence using magnetic quadrupoles or electrostatic lenses to maintain beam.
e. **Beam Extraction from Ion Source** High voltage applied between the ion source and an extraction electrode creates a stable, well-formed beam with controlled energy and emittance for downstream processes.
f. **Beam in Drift Space** In drift spaces, the beam propagates without external forces, with size increasing in each direction due to space charge effects. Periodic focusing is necessary to preserve beam quality.

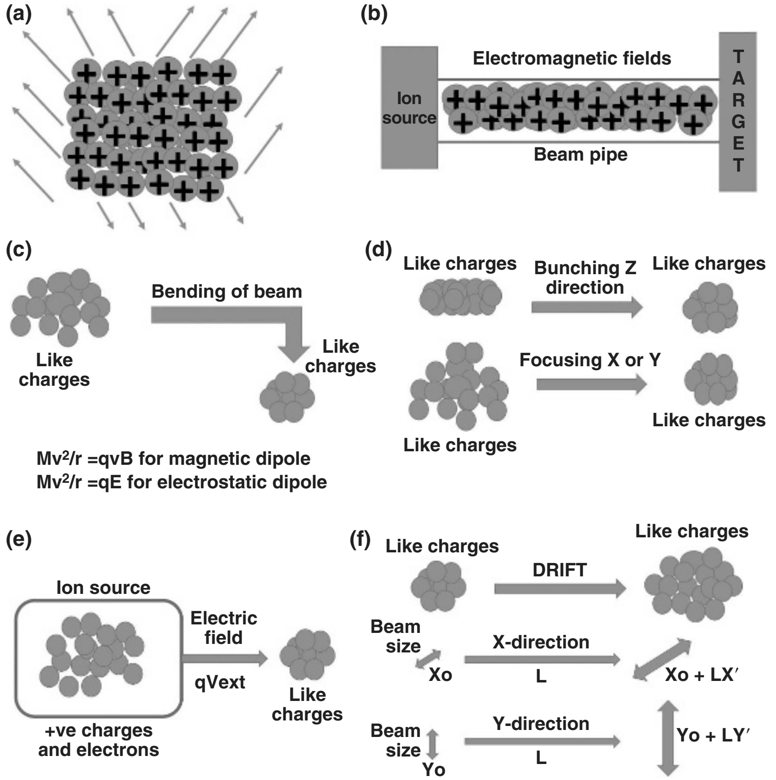

Figure 1.6 (a) Collection of like charges, which never wants to stay together. (b) Beam transport from ion source to target under the effect of electromagnetic fields. (c) Beam analysis using magnetic or electric fields as per the Lorentz force formula due to magnetic or electric fields. (d) Beam bunching and focusing in the longitudinal and transverse directions showing control of beam size by these operations. (e) Beam extraction from source using voltage V_{ext} so that particle energy becomes qV_{ext}. (f) Beam along a drift space showing beam blow up in both transverse planes.

1.13 Numerical Problems

1. Calculate the resolution of an optical microscope based on the wavelength of visible light 500 nm. Then, calculate the resolution of a particle accelerator using the de Broglie wavelength of an electron with a kinetic energy of 10 MeV. Using the electron's rest mass, compare the resolutions of the two devices and determine the ratio of the microscope's resolution to the accelerator's resolution. Finally, explain the significance of this comparison.
2. A Cockcroft–Walton type voltage multiplier has eight stages, each with a capacitance of 0.06 μF. The transformer generates a secondary voltage of 100 kV at 150 Hz frequency. To supply a load current of 10 mA, calculate the percentage ripple, regulation, and the optimal number of stages for minimum voltage drop.
3. Calculate the frequency and radius of a cyclotron for 15 MeV deuterons with a magnetic field of 1 T. The potential difference across the dees is 20 kV. How many turns will the deuteron take to reach a final energy of 15 MeV?

4. A cyclotron accelerates protons by 500 V each time they cross the gap between dees, with an outer radius of 400 mm and a magnetic field of 0.8 T. Calculate the cyclotron frequency of the protons, the maximum kinetic energy, and the total number of revolutions the protons will make.
5. Calculate the **cyclotron frequency** for the following cases:
   (a) A proton is placed in a magnetic field of 1 T (nonrelativistic case).
   (b) A proton with a kinetic energy of 200 MeV is subjected to a magnetic field of 8 T.
   (c) An Ar^{16+} ion, with a kinetic energy of 2 GeV, is placed in a magnetic field of 4 T.
6. Calculate the upper limit for the kinetic energy of protons attainable in a cyclotron with a 1.5 T magnetic field and a 1 m radius.
7. What is the terminal potential required in a Tandem accelerator to obtain 100 MeV proton, carbon, nickel, and gold ion beams? Assume the injection energy to the Tandem accelerator is 200 keV.
8. What is the maximum energy obtained from a Tandem accelerator with a maximum terminal potential of 15 MV for proton, carbon, nickel, and gold ion beams?
9. A synchrotron is designed to accelerate protons to an energy of 2 GeV. It contains 40 dipole magnets, each with an effective length of 2 m, and operates with a magnetic field strength of 1.5 T. The synchrotron has a filling factor K_F of 0.6. Calculate the following: the magnetic rigidity $B\rho$ of the synchrotron, the integrated dipole strength $B \cdot l$ for each dipole magnet, the circumference C of the synchrotron ring, the bending radius ρ of the particle trajectory within the dipole magnets and the required RF for a harmonic number $h = 100$. Assume the particles are moving at nearly the speed of light.
10. A synchrocyclotron is designed to accelerate protons to an energy of 200 MeV. The magnetic field at the proton's orbit is 1.8 T, and the orbit radius is 0.7 m. Calculate the following: the relativistic factor γ of the proton, the final velocity of the proton, the required RF for the synchrocyclotron at this energy, and the time taken for the proton to complete one revolution.
11. A betatron is designed to accelerate electrons to high energies. The average magnetic field $\langle B \rangle$ over the area enclosed by the electron's orbit is changing at a rate of $\frac{d\langle B \rangle}{dt} =$ 50 T/s. The radius of the electron's orbit is 1 m, and the magnetic field at the orbit B_{orbit} is half the average magnetic field $\langle B \rangle$. Calculate the following: the induced electric field E around the electron's orbit, the force F exerted on the electron by the induced electric field, the momentum p of the electron after 1 μs ($t = 10^{-6}$ s), and the magnetic field at the orbit B_{orbit} if the average magnetic field $\langle B \rangle$ is 1 T.
12. An electron is accelerated in a nonscaling FFAG accelerator, where the magnetic field at the reference orbit is 0.5 T with a reference radius of 1.0 m and a magnetic field index of 0.3. The electron is injected with an energy of 1 MeV and accelerated to 5 MeV. Calculate the orbit radius of the electron at 5 MeV and the revolution times at both 1 and 5 MeV, assuming relativistic conditions for the electron's velocity.
13. Calculate the corresponding laser intensities I_0 [W/cm^2] for a laser with a wavelength of 1 μm and normalized vector potential a_0 for the following values: 0.02, 0.1, 0.5, and 1.0.

14. Calculate the plasma wavelength λ_p in micrometers (μm) and the plasma electric field E_p in GV/m for a plasma with an electron density $n_e = 5 \times 10^{18}\,\text{cm}^{-3}$ and a relative density perturbation $\frac{\delta n_e}{n_e} = 0.1$.

15. Calculate the dose rate for a Cesium-137 source with an activity of $A = 6\,\text{Ci}$ at a distance of 1 m using the inverse square law. The gamma constant is given as $\Gamma = 0.33\,\text{mSv} \cdot \text{m}^2/\text{Ci} \cdot \text{h}$. Then, calculate the dose rate after applying 1 cm of lead shielding, where the linear attenuation coefficient for lead is $\mu = 0.55\,\text{cm}^{-1}$, using the exponential attenuation formula. Finally, calculate the total dose received after of exposure by multiplying the dose rate after shielding with the exposure time.