Guido De Philippis

Regularity of Optimal Transport Maps and Applications

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Introduction

This thesis is devoted to the regularity of optimal transport maps. We provide new results on this problem and some applications. This is part of the work done by the author during his PhD studies. Other papers written during the PhD studies and not completely related to this topic are summarized in the second part of the introduction.

1. Regularity of optimal transport maps and applications

Monge optimal transportation problem goes back to 1781 and it can be stated as follows:
Given two probability densities ρ_1 and ρ_2 on $\mathbb{R}^n$ (originally representing the height of a pile of soil and the depth of an excavation), let us look for a map T moving ρ_1 onto ρ_2, *i.e.* such that[1]

$$\int_{T^{-1}(A)} \rho_1(x)\,dx = \int_A \rho_2(y)\,dy \qquad \text{for all Borel sets } A, \tag{1}$$

and minimizing the total cost of such process:

$$\int c(x, T(x))\rho_1(x)dx = \inf\left\{\int c(x, S(x))\rho_1(x)\,dx \,:\, S \text{ satisfies (1)}\right\}. \tag{2}$$

Here $c(x, y)$ represent the "cost" of moving a unit of mass from x to y (the original Monge's formulation the cost $c(x, y)$ was given by $|x - y|$).

Conditions for the existence of an optimal map T are by now well understood (and summarized without pretending to be aexhaustive in Chapter 1, see [95, Chapter 10] for a more recent account of the theory).

Once the existence of an optimal map has been established a natural question is about its *regularity*. Informally the question can be stated as follows:
Given two smooth densities, ρ_1 and ρ_2 supported on good sets, it is true the T is smooth?

[1] From the mathematical point of view we are requiring that $T_\sharp(\rho_1\mathscr{L}^n) = \rho_2\mathscr{L}^n$, see Chapter 1.

Or, somehow more precisely, one can investigate how much is the "gain" in regularity from the densities to T. As we will see in a moment, a natural guess is that T should have "one derivative" more than ρ_1 and ρ_2.

To start investigating regularity, notice that (1) can be re-written as

$$|\det \nabla T(x)| = \frac{\rho_1(x)}{\rho_2(T(x)))}, \tag{3}$$

which turns out to be a very degenerate first order PDE. As we already said, the above equation could lead to the guess that T has one derivative more than the densities. Notice however that the above equation is satisfied by *every* map which satisfies (1). Thus, by simple examples, we cannot expect solutions of (3) to be well-behaved. Indeed, consider for instance the case in which $\rho_1 = \mathbf{1}_A$ and $\rho_2 = \mathbf{1}_B$ with A and B smooth open sets. If we right (respectively left) compose T with a map S satisfying $\det \nabla S = 1$ and $S(A) = A$ (resp. $S(B) = B$) we still obtain a solution of (3) which is no more regular than S.

It is at this point that condition (2) comes into play. To see how, let us focus on the *quadratic case*, $c(x, y) = |x - y|^2/2$. In this case Brenier Theorem 1.8, ensures that the optimal T is given by the gradient of a convex function, $T = \nabla u$. Plugging this information into (3) we obtain that u solves the following *Monge-Ampère* equation

$$\det \nabla^2 u(x) = \frac{\rho_1(x)}{\rho_2(\nabla u(x)))}. \tag{4}$$

In this way we have obtained a (degenerate) elliptic second order PDE, and there is hope to obtain regularity of $T = \nabla u$ from the regularity of the densities.[2] In spite of the above discussion, also equation (4) it is not enough to ensure regularity of u. A simple example is given by the case in which the support of the first density is connected while the support of the second is not. Indeed, since by (1) it follows easily that

$$\overline{T(\operatorname{spt} \rho_1)} = \operatorname{spt} \rho_2,$$

[2] One should compare this with the following fact: there is no hope to get regularity of a vector field $\boldsymbol{v}$ satisfying

$$\nabla \cdot \boldsymbol{v} = 0,$$

while if we add the additional condition $\boldsymbol{v} = \nabla u$ we obtain the Laplace equation

$$\Delta v = 0.$$

we immediately see that, even if the densities are smooth on their supports, T has to be discontinuous (cp. Example 1.16). It was noticed by Caffarelli, [21], that the right assumption to be made on the support of ρ_2 is *convexity*. In this case any solution of (4) arising from the optimal transportation problem turns out to be a strictly convex *Aleksandrov solution* to the Monge-Ampère equation[3]

$$\det D^2 u = \frac{\rho_1(x)}{\rho_2(\nabla u(x)))} \qquad \text{on } \mathrm{Int}(\mathrm{spt}\,\rho_1). \tag{5}$$

As a consequence, under the previous assumptions, we can translate any regularity results for Aleksandrov solutions to the Monge-Ampère equation to solution to the optimal transport problem. In particular, by the theory developed in [18, 19, 20, 89] (see also [66, Chapter 17]) we have the following (see Chapter 2 for a more precise discussion):

- If ρ_1 and ρ_2 are bounded away from zero and infinity on their support and spt ρ_2 is convex, then $u \in C^{1,\alpha}_{\mathrm{loc}}$ (and hence $T \in C^{\alpha}_{\mathrm{loc}}$).
- If, in addition, ρ_1 and ρ_2 are continuous, then $T \in W^{2,p}_{\mathrm{loc}}$ for every $p \in [1, \infty)$.
- If ρ_1 and ρ_2 are $C^{k,\beta}$ and, again, spt ρ_2 is convex, then $T \in C^{k+2,\beta}_{\mathrm{loc}}$.

A natural question which was left open by the above theory is the Sobolev regularity of T under the only assumptions that ρ_1 and ρ_2 are bounded away from zero and infinity on their support and spt ρ_2 is convex. In [93], Wang shows with a family of counterexamples that the best one can expect is $T \in W^{1,1+\varepsilon}$ with $\varepsilon = \varepsilon(n, \lambda)$, where λ is the "pinching" $\|\log(\rho_1/\rho_2(\nabla u))\|_\infty$, see Example 2.21.

Apart from being a very natural question from the PDE point of view, Sobolev regularity of optimal transport maps (or equivalently of Aleksandrov solutions to the Monge-Ampère equation) has a relevant application to the study of the semigeostrophic system, as was pointed out by Ambrosio in [4]. This is a system of equations arising in study of large oceanic and atmospheric flows. Referring to Chapter 5 for a more accurate discussion we recall here that the system can be written, after a

[3] This kind of solutions have been introduced by Aleksandrov in the study of the Minkowski Problem: given a function $\kappa : \mathbb{S}^{n-1} \to [0, \infty)$ find a convex body $\mathcal{K}$ such that the Gauss curvature of its boundary is given by $\kappa \circ \nu_{\partial\mathcal{K}}$. All the results of Chapters 2, 3, 4, apply to this problem as well.

suitable change of variable, as

$$\begin{cases} \partial_t \nabla P_t + (\boldsymbol{u}_t \cdot \nabla)\nabla P_t = J(\nabla P_t - x) & \text{in } \Omega \times (0, +\infty) \\ \nabla \cdot \boldsymbol{u}_t = 0 & \text{in } \Omega \times (0, +\infty) \\ \boldsymbol{u}_t \cdot \nu_\Omega = 0 & \text{in } \partial\Omega \times (0, +\infty) \\ P_0 = P^0 & \text{in } \Omega, \end{cases} \tag{6}$$

where Ω is an open, bounded and convex subset of $\mathbb{R}^3$ and

$$J := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We look for solutions P_t which are *convex* for every t (this ansatz is based on the Cullen stability principle [34, Section 3.2]). If we consider the measure[4] $\rho_t = (\nabla u)_\sharp \mathscr{L}^3_\Omega$, then ρ_t solves (formally) the following continuity type equation

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (\boldsymbol{\mathcal{U}}_t \rho_t) = 0 \\ \boldsymbol{\mathcal{U}}_t(x) = J(x - \nabla P_t^*(x)) \\ (\nabla P_t^*)_\sharp \rho_t = \mathscr{L}^3_\Omega, \end{cases} \tag{7}$$

where P_t^* is the convex conjugate of P_t. Even if the velocity field $\boldsymbol{\mathcal{U}}_t$ is coupled with the density through a highly non-linear equation, existence of (distributional) solutions of (7) can be obtained under very mild conditions on the initial densities $\rho_0 = (\nabla P_0)_\sharp \mathscr{L}^3_\Omega$, [13]. Given a solution of (7) we can formally obtain a solution to (6) by taking $P_t = (P_t^*)^*$ and

$$\boldsymbol{u}_t(x) := [\partial_t \nabla P_t^*](\nabla P_t(x)) + [\nabla^2 P_t^*](\nabla P_t(x)) J(\nabla P_t(x) - x). \tag{8}$$

To give a meaning to the above velocity field we have to understand the regularity of $\nabla^2 P_t^*$ where P_t^* satisfies $(\nabla P_t^*)_\sharp \rho_t = \mathscr{L}^3_\Omega$. Notice that the only condition we get for free is that $\boldsymbol{\mathcal{U}}_t$ has zero divergence. In particular, if the initial density ρ_0 is bounded away from zero and infinity, the same it is true (with the same bounds) for ρ_t. It is then natural to study the $W^{2,1}$ regularity of solutions of (5) under the only assumption that the right hand side is bounded between two positive constants. This is done in

[4] With $\mathscr{L}^3_\Omega$ we denote the normalized Lebesgue measure restricted to Ω:

$$\mathscr{L}^3_\Omega := \frac{1}{\mathscr{L}^3(\Omega)} \mathscr{L}^3 \llcorner \Omega$$

.

Chapters 3 and 4 (based on [40, 41] in collaboration with Alessio Figalli, and on [44] in collaboration with Alessio Figalli and Ovidiu Savin) while in Chapter 5 (based on [5, 6] in collaboration with Luigi Ambrosio, Maria Colombo and Alessio Figalli) we study the applications of this results to the semigeostrophic system.

Finally we came back to the regularity of solutions of (2) with a general cost function c, referring to Section 1.3 for a more complete discussion. In this case, apart from the obstruction given by the geometry of the target domain (as in the quadratic cost case) it has been shown in [80, 78] that a structural condition on the cost function, the so called *MTW-condition*, is needed in order to ensure the smoothness of the optimal transport map. In particular if the above condition does not hold it is possible to construct two smooth densities such that the optimal map between them is even discontinuous.

In spite of this, one can try to understand how large can be the set of discontinuity points of optimal maps between two smooth densities for a generic smooth cost c. In Chapter 6 (based on [43] in collaboration with Alessio Figalli), we will show that, under very mild assumptions on the cost c (essentially the one needed in order to get existence of optimal maps), there exist two closed and Lebesgue negligible sets Σ_1 and Σ_2 such that

$$T : \operatorname{spt}\rho_1 \setminus \Sigma_1 \to \operatorname{spt}\rho_2 \setminus \Sigma_2$$

is a smooth diffeomorphism. A similar result holds true also in the case of optimal transportation on Riemannian manifolds with cost $c = d^2/2$. Up to now similar results were known only in the case of quadratic cost when the support of the target density is not convex, [52, 55]. We remark here that in this case the obstruction to regularity is given only by the geometry of the domain, while in the case of a generic cost function c we have to face the possible failure of the MTW condition at every point. Thus, to achieve the proof of our result, we have to use a completely different strategy.

We conclude this first part of the introduction with a short summary of each chapter of the thesis (more details are given at the beginning of each chapter):

- **Chapter 1.** In this Chapter we briefly recall the general theory of optimal transportation, with a particular focus on the case of quadratic cost in $\mathbb{R}^n$. We also show how to pass from solutions of the Monge-Ampère equation given by the optimal transportation to Aleksandrov solutions to the Monge-Ampère equation in case the support of the target density is convex. Finally in the last Section we address the case of a general cost function.

- **Chapter 2.** We start the study of the regularity of Aleksandrov solutions to the Monge-Ampère equation, in particular we give a complete proof of Caffarelli's $C^{1,\alpha}$ regularity theorem.
- **Chapter 3.** We start investigating the $W^{2,1}$ regularity of Aleksandrov solutions to the Monge-Ampère equation. We give a complete proof of the results in [40], where we show that $D^2u \in L \log L$. Then, following the subsequent paper [44], we show how the above estimate can be improved to $D^2u \in L^{1+\varepsilon}$ for some small $\varepsilon > 0$. We also give a short proof of the above mentioned Caffarelli $W^{2,p}$ estimates.
- **Chapter 4.** Here, following [41], we show the (somehow surprising) stability in the *strong* $W^{2,1}$ topology of Aleksandrov solutions with respect to the L^1 convergence of the right-hand sides.
- **Chapter 5.** In this Chapter, based on [5, 6], we apply the results of the previous chapters to show the existence of a distributional solution to the semigeostrophic system (6) in the 2-dimensional periodic case and in the case of a bounded convex 3-dimensional domain Ω. In the latter case we have to impose a suitable decay assumption on the initial density $\rho_0 = (\nabla P_0)_\sharp \mathscr{L}^3_\Omega$.
- **Chapter 6.** Here we report the partial regularity theorems for solutions of the optimal transport problem for a general cost function c proved in [43].

2. Other papers

In this second part of the introduction we give a short summary of the additional research made during the PhD studies, only vaguely related to the theme of the thesis. We briefly report the results obtained and we refer to the original papers for a more complete treatment of the problem and the relevant literature.

1. Γ-convergence of non-local perimeter

In [29] Caffarelli-Roquejoffre and Savin introduced the following notion of *non-local* perimeter of a set E relative of an open set Ω:

$$\mathcal{J}_s(E,\Omega) = \int_{E\cap\Omega}\int_{E^c\cap\Omega}\frac{dxdy}{|x-y|^{n+s}} + \int_{E\cap\Omega}\int_{E^c\cap\Omega^c}\frac{dxdy}{|x-y|^{n+s}} + \int_{E\cap\Omega^c}\int_{E^c\cap\Omega}\frac{dxdy}{|x-y|^{n+s}},$$

and study the regularity of local minimizers of it. This functional naturally arises in the study of phase-transitions with a non-local interaction term, see the nice survey [63] and reference therein for an updated account of the theory.

In [10], in collaboration with Luigi Ambrosio and Luca Martinazzi, we show the Γ-convergence of the functional $(1-s)\mathcal{J}_s(\cdot,\Omega)$ to the classical De Giorgi perimeter $\omega_{n-1}P(\cdot,\Omega)$ with respect to the topology of locally L^1 convergence of sets (a similar earlier result has been obtained in [30] for the convergence of local minimizers of the functionals). We also show equicoercivity of the functionals. More precisely we prove:

Theorem. *Let $s_i \uparrow 1$, then the following statements hold:*

(i) (Equicoercivity). *Assume that E_i are measurable sets satisfying*

$$\sup_{i\in\mathbb{N}}(1-s_i)\mathcal{J}^1_{s_i}(E_i,\Omega') < \infty \qquad \forall\Omega' \Subset \Omega.$$

Then $\{E_i\}_{i\in\mathbb{N}}$ is relatively compact in $L^1_{\mathrm{loc}}(\Omega)$ and any limit point E has locally finite perimeter in Ω.

(ii) (Γ-convergence). *For every measurable set $E \subset \mathbb{R}^n$ we have*

$$\Gamma - \lim_{s\uparrow 1}(1-s)\mathcal{J}_s(E,\Omega) = \omega_{n-1}P(E,\Omega).$$

with respect to the the L^1_{loc} convergence of the corresponding characteristic functions in $\mathbb{R}^n$.

(iii) (Convergence of local minimizers). *Assume that E_i are local minimizers of $\mathcal{J}_{s_i}(\cdot,\Omega)$, and $E_i \to E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then*

$$\limsup_{i\to\infty}(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega') < +\infty \qquad \forall\Omega' \Subset \Omega,$$

E is a local minimizer of $P(\cdot,\Omega)$ and $(1-s_i)\mathcal{J}_{s_i}(E_i,\Omega') \to \omega_{n-1}P(E,\Omega')$ whenever $\Omega' \Subset \Omega$ and $P(E,\partial\Omega')=0$.

2. Sobolev regularity of optimal transport map and differential inclusions

In [9], written in collaboration with Luigi Ambrosio and Bernd Kirchheim, we started the investigation of the Sobolev regularity of (stricly convex) Aleksandrov solution to the Monge-Ampé re equation. More precisely we show that in the 2-dimensional case the Sobolev regularity of optimal transport maps is *equivalent* to the rigidity of a partial differential inclusion for Lipschitz maps (see [74, 84] for nice surveys on partial differential inclusions). Referring to the paper for more details, let us define the set of "admissible" gradients

$$\mathcal{A} := \Big\{ M \in \mathrm{Sym}^{2\times 2} : \ \|M\| \le 1, \ (\lambda+1)|\mathrm{Trace}(M)| \le (1-\lambda)(1+\det(M)) \Big\}, \tag{9}$$