
Asymptotics in Dynamics, Geometry and PDEs; Generalized Borel Summation vol. II

edited by
O. Costin, F. Faÿvet,
F. Menous, D. Sauzin



EDIZIONI
DELLA
NORMALE

12**

CRM
SERIES



Centro
di Ricerca
Matematica
Ennio De Giorgi

Ovidiu Costin
Mathematics Department
The Ohio State University
231 W. 18th Avenue
Columbus, Ohio 43210, USA
costin@math.ohio-state.edu

Frédéric Fauvet
Département de mathématiques - IRMA
Université de Strasbourg
7, rue Descartes
67084 Strasbourg CEDEX, France
fauvet@math.u-strasbg.fr

Frédéric Menous
Département de mathématiques, Bât. 425
Université Paris-Sud
91405 Orsay CEDEX, France
Frederic.MENOUS@math.u-psud.fr

David Sauzin
CNRS Paris
and
Scuola Normale Superiore
Piazza dei Cavalieri 7
56126 Pisa, Italia
sauzin@imcce.fr
david.sauzin@sns.it

Asymptotics in Dynamics, Geometry and PDEs; Generalized Borel Summation vol. II

edited by
O. Costin, F. Fauvet,
F. Menous, D. Sauzin



EDIZIONI
DELLA
NORMALE

© 2011 Scuola Normale Superiore Pisa

ISBN: 978-88-7642-376-5

e-ISBN: 978-88-7642-377-2

Contents

Introduction	xi
Authors' affiliations	xv
Christian Bogner and Stefan Weinzierl	
Feynman graphs in perturbative quantum field theory	1
1 Introduction	1
2 Perturbation theory	2
3 Multi-loop integrals	6
4 Periods	7
5 A theorem on Feynman integrals	8
6 Sector decomposition	10
7 Hironaka's polyhedra game	12
8 Shuffle algebras	13
9 Multiple polylogarithms	17
10 From Feynman integrals to multiple polylogarithms . . .	19
11 Conclusions	22
References	23
Jean Ecalle	
with computational assistance from S. Carr	
The flexion structure and dimorphy: flexion units, singulators, generators, and the enumeration of multizeta irreducibles	27
1 Introduction and reminders	30
1.1 Multizetas and dimorphy	30
1.2 From scalars to generating series	32
1.3 ARI//GARI and its dimorphic substructures . . .	35
1.4 Flexion units, singulators, double symmetries . .	36

1.5	Enumeration of multizeta irreducibles	37
1.6	Canonical irreducibles and perinomial algebra	38
1.7	Purpose of the present survey	38
2	Basic dimorphic algebras	40
2.1	Basic operations	40
2.2	The algebra ARI and its group $GARI$	45
2.3	Action of the basic involution $swap$	48
2.4	Straight symmetries and subsymmetries	49
2.5	Main subalgebras	52
2.6	Main subgroups	53
2.7	The dimorphic algebra $ARI^{\underline{al}/\underline{al}} \subset ARI^{al/al}$	53
2.8	The dimorphic group $GARI^{\underline{as}/\underline{as}} \subset GARI^{as/as}$	54
3	Flexion units and twisted symmetries	54
3.1	The free monogenous flexion algebra $Flex(E)$	54
3.2	Flexion units	57
3.3	Unit-generated algebras $Flex(E)$	61
3.4	Twisted symmetries and subsymmetries in universal mode	63
3.5	Twisted symmetries and subsymmetries in polar mode	67
4	Flexion units and dimorphic bimoulds	70
4.1	Remarkable substructures of $Flex(E)$	70
4.2	The secondary bimoulds $ess^\bullet$ and $esz^\bullet$	79
4.3	The related primary bimoulds $es^\bullet$ and $ez^\bullet$	87
4.4	Some basic bimould identities	88
4.5	Trigonometric and bitrigonometric bimoulds	89
4.6	Dimorphic isomorphisms in universal mode	94
4.7	Dimorphic isomorphisms in polar mode	95
5	Singulators, singulands, singulates	99
5.1	Some heuristics. Double symmetries and imparity	99
5.2	Universal singulators $senk(ess^\bullet)$ and $seng(es^\bullet)$	101
5.3	Properties of the universal singulators	102
5.4	Polar singulators: description and properties	104
5.5	Simple polar singulators	105
5.6	Composite polar singulators	105
5.7	From $\underline{al}/\underline{al}$ to $\underline{al}/\underline{il}$. Nature of the singularities	106
6	A natural basis for $ALIL \subset ARI^{\underline{al}/\underline{il}}$	107
6.1	Singulation-desingulation: the general scheme	107
6.2	Singulation-desingulation up to length 2	111
6.3	Singulation-desingulation up to length 4	112
6.4	Singulation-desingulation up to length 6	112
6.5	The basis $lama^\bullet/lami^\bullet$	116

6.6	The basis $loma^\bullet/lomi^\bullet$	116
6.7	The basis $luma^\bullet/lumi^\bullet$	117
6.8	Arithmetical vs analytic smoothness	118
6.9	Singulator kernels and “wandering” bialternals	119
7	A conjectural basis for $ALAL \subset ARI^{\underline{al}/\underline{al}}$. The three series of bialternals	120
7.1	Basic bialternals: the enumeration problem	120
7.2	The regular bialternals: $ekma, doma$	120
7.3	The irregular bialternals: $carma$	121
7.4	Main differences between regular and irregular bialternals	121
7.5	The <i>pre-doma</i> potentials	123
7.6	The <i>pre-carma</i> potentials	124
7.7	Construction of the <i>carma</i> bialternals	126
7.8	Alternative approach	127
7.9	The global bialternal ideal and the universal ‘restoration’ mechanism	129
8	The enumeration of bialternals. Conjectures and computational evidence	130
8.1	Primary, sesquary, secondary algebras	130
8.2	The ‘factor’ algebra $EKMA$ and its subalgebra $DOMA$	132
8.3	The ‘factor’ algebra $CARMA$	133
8.4	The total algebra of bialternals $ALAL$ and the original BK-conjecture	133
8.5	The factor algebras and our sharper conjectures	133
8.6	Cell dimensions for $ALAL$	135
8.7	Cell dimensions for $EKMA$	135
8.8	Cell dimensions for $DOMA$	136
8.9	Cell dimensions for $CARMA$	136
8.10	Computational checks (Sarah Carr)	137
9	Canonical irreducibles and perinomial algebra	139
9.1	The general scheme	139
9.2	Arithmetical criteria	144
9.3	Functional criteria	144
9.4	Notions of perinomial algebra	147
9.5	The all-encoding perinomial mould $peri^\bullet$	149
9.6	A glimpse of perinomial splendour	150
10	Provisional conclusion	152
10.1	Arithmetical and functional dimorphy	152
10.2	Moulds and bimoulds. The flexion structure	154

10.3	<i>ARI/GARI</i> and the handling of double symmetries	158
10.4	What has already been achieved	160
10.5	Looking ahead: what is within reach and what beckons from afar	164
11	Complements	165
11.1	Origin of the flexion structure	165
11.2	From simple to double symmetries. The <i>scramble</i> transform	167
11.3	The bialternal tessellation bimould	168
11.4	Polar, trigonometric, bitrigonometric symmetries	171
11.5	The separative algebras $Inter(Qi_c)$ and $Exter(Qi_c)$	175
11.6	Multizeta cleansing: elimination of unit weights	180
11.7	Multizeta cleansing: elimination of odd degrees	189
11.8	$GARI_{se}$ and the two separation lemmas	192
11.9	Bisymmetry of $ess^\bullet$: conceptual proof	193
11.10	Bisymmetry of $ess^\bullet$: combinatorial proof	195
12	Tables, index, references	198
12.1	Table 1: basis for $Flex(E)$	198
12.2	Table 2: basis for $Flexin(E)$	202
12.3	Table 3: basis for $Flexinn(E)$	202
12.4	Table 4: the universal bimould $ess^\bullet$	205
12.5	Table 5: the universal bimould $esz_\sigma^\bullet$	206
12.6	Table 6: the bitrigonometric bimould $taal^\bullet/tiil^\bullet$	207
12.7	Index of terms and notations	207
	References	210

Augustin Fruchard and Reinhard Schäfke

	On the parametric resurgence for a certain singularly perturbed linear differential equation of second order	213
1	Introduction	213
2	The distinguished solutions	216
3	Stokes relations for the functions	221
4	Behavior near the turning points	223
5	The residue	226
6	Stokes relations for the wronskians	228
7	Stokes relations for the factors of the wronskians	229
8	Resurgence of the series $\widehat{r}(\varepsilon)$ and $\widehat{\gamma}^\pm$	232
9	Resurgence of the WKB solution	238
10	Remarks and Perspectives	240
	References	241

Shingo Kamimoto, Takahiro Kawai, Tatsuya Koike and Yoshitsugu Takei	
On a Schrödinger equation with a merging pair of a simple pole and a simple turning point — Alien calculus of WKB solutions through microlocal analysis	245
References	253
Yoshitsugu Takei	
On the turning point problem for instanton-type solutions of Painlevé equations	255
1 Background and main results	255
2 Transformation near a double turning point	261
2.1 Exact WKB theoretic structure of $(P_{\text{II,deg}})$	261
2.2 Transformation theory to $(P_{\text{II,deg}})$ near a double turning point	265
3 Transformation near a simple pole	269
References	273

Introduction

In the last three decades or so, important questions in distinct areas of mathematics such as the local analytic dynamics, the study of analytic partial differential equations, the classification of geometric structures (*e.g.* moduli for holomorphic foliations), or the semi-classical analysis of Schrödinger equation have necessitated in a crucial way to handle delicate asymptotics, with the formal series involved being generally divergent, displaying specific growth patterns for their coefficients, namely of Gevrey type. The modern study of Gevrey asymptotics, for questions originating in geometry or analysis, goes together with an investigation of rich underlying algebraic concepts, revealed by the application of Borel resummation techniques.

Specifically, the study of the Stokes phenomenon has had spectacular recent applications in questions of integrability, in dynamics and PDEs. Some generalized form of Borel summation has been developed to handle the relevant structured expansions – named transseries – which mix series, exponentials and logarithms; these formal objects are in fact ubiquitous in special function theory since the 19th century.

Perturbative Quantum Field Theory is also a domain where recent advances have been obtained, for series and transseries which are of a totally different origin from the ones met in local dynamics and yet display the same sort of phenomena with, strikingly, the very same underlying algebraic objects.

Hopf algebras, *e.g.* with occurrences of shuffle and quasishuffle products that are important themes in the algebraic combinatorics community, appear now natural and useful in local dynamics as well as in pQFT. One common thread in many of the important advances for these questions is the concept of resurgence, which has triggered substantial progress in various areas in the near past.

An international conference took place on October 12th – October 16th, 2009, in the Centro di Ricerca Matematica Ennio De Giorgi, in Pisa, to highlight recent achievements along these ideas.

Here is a complete list of the lectures delivered during this event:

Carl Bender, *Complex dynamical systems*

Filippo Bracci, *One resonant biholomorphisms and applications to quasi-parabolic germs*

David Broadhurst, *Multiple zeta values in quantum field theory*

Jean Ecalle, *Four recent advances in resummation and resurgence theory*

Adam Epstein, *Limits of quadratic rational maps with degenerate parabolic fixed points of multiplier $e^{2\pi i q} \rightarrow 1$*

G rard Iooss, *On the existence of quasipattern solutions of the Swift-Hohenberg equation and of the Rayleigh-Benard convection problem*

Shingo Kamimoto, *On a Schr dinger operator with a merging pair of a simple pole and a simple turning point, I: WKB theoretic transformation to the canonical form*

Tatsuya Koike, *On a Schr dinger operator with a merging pair of a simple pole and a simple turning point, II: Computation of Voros coefficients and its consequence*

Dirk Kreimer, *An analysis of Dyson Schwinger equations using Hopf algebras*

Joel Lebowitz, *Time Asymptotic Behavior of Schr dinger Equation of Model Atomic Systems with Periodic Forcings: To Ionize or Not*

Carlos Matheus, *Multilinear estimates for the 2D and 3D Zakharov-Rubenchik systems*

Emmanuel Paul, *Moduli space of foliations and curves defined by a generic function*

Jasmin Raissy, *Torus actions in the normalization problem*

Javier Ribon, *Multi-summability of unfoldings of tangent to the identity diffeomorphisms*

Reinhard Sch fke, *An analytic proof of parametric resurgence for some second order linear equations*

Mitsuhiro Shishikura, *Invariant sets for irrationally indifferent fixed points of holomorphic functions*

Harris J. Silverstone, *Kramers-Langer-modified radial JWKB equations and Borel summability*

Yoshitsugu Takei, *On the turning point problem for instanton-type solutions of (higher order) Painlev  equations*

Saleh Tanveer, *Borel Summability methods applied to PDE initial value problems*

Jean-Yves Thibon, *Noncommutative symmetric functions and combinatorial Hopf algebras*

Valerio Toledano Laredo, *Stokes factors and multilogarithms*

Stefan Weinzierl, *Feynman graphs in perturbative quantum field theory*
Sergei Yakovenko, *Deficity of intersections between trajectories of ODEs and algebraic hypersurfaces*
Michael Yampolsky, *Geometric properties of a parabolic renormalization fixed point*

The present volume, together with a first one already published in the same collection, contains five contributions of invited speakers at this conference, reflecting some of the leading themes outlined above.

We express our deep gratitude to the staffs of the Scuola Normale Superiore di Pisa and of the CRM Ennio de Giorgi, in particular to the Director of the CRM, Professor Mariano Giaquinta, for their dedicated support in the preparation of this meeting; all participants could thus benefit of the wonderful and stimulating atmosphere in these institutions and around Piazza dei Cavalieri. We are also very grateful for the possibility to publish these two volumes in the CRM series. We acknowledge with thankfulness the support of the CRM, of the ANR project “Resonances”, of Université Paris 11 and of the Gruppo di Ricerca Europeo Franco-Italiano: Fisica e Matematica, with also many thanks to Professor Jean-Pierre Ramis. All our recognition for the members of the Scientific Board for the conference: Professors Louis Boutet de Monvel (Univ. Paris 6), Dominique Cerveau (Univ. Rennes), Takahiro Kawai (RIMS, Kyoto) and Stefano Marmi (SNS Pisa).

Pisa, June 2011

Ovidiu Costin, Frédéric Fauvet,
Frédéric Menous and David Sauzin

Authors' affiliations

CHRISTIAN BOGNER – Institut für Theoretische Physik E, RWTH Aachen,
D - 52056 Aachen, Germany
bogner@physik.rwth-aachen.de

JEAN ECALLE – Département de mathématiques, Bât. 425 Université
Paris-Sud, 91405 Orsay Cedex - France
Jean.Ecalle@math.u-psud.fr

AUGUSTIN FRUCHARD – LMIA, EA 3993, Université de haute Alsace,
4, rue des Frères-Lumière, 68093 Mulhouse cedex, France
augustin.fruchard@uha.fr

SHINGO KAMIMOTO – Graduate School of Mathematical Sciences, Uni-
versity of Tokyo, Tokyo, 153-8914, Japan
kamimoto@ms.u-tokyo.ac.jp

TAKAHIRO KAWAI – Research Institute for Mathematical Sciences, Ky-
oto University, Kyoto, 606-8502, Japan

TATSUYA KOIKE – Department of Mathematics, Graduate School of Sci-
ence, Kobe University, Kobe, 657-8501, Japan
koike@math.kobe-u.ac.jp

REINHARD SCHÄFKE – IRMA, UMR 7501, Université de Strasbourg et
CNRS, 7, rue René-Descartes, 67084 Strasbourg cedex, France
schaecke@math.u-strasbg.fr

YOSHITSUGU TAKEI – Research Institute for Mathematical Sciences,
Kyoto University, Kyoto, 606-8502, Japan
takei@kurims.kyoto-u.ac.jp

STEFAN WEINZIERL – Institut für Physik, Universität Mainz, D - 55099
Mainz, Germany
stefanw@thep.physik.uni-mainz.de