

Introduction to the Geometrical Foundations of General Relativity

A Compendium of Tensor Calculus for Physicists

Bernd Wichmann

2nd edition

Bernd Wichmann, Kufstein, Austria, bernd47@kufnet.at, www.tensor-calculus.com

Text: © Copyright by Bernd Wichmann

Graphics: © Copyright by Bernd Wichmann

All rights reserved

1st edition 2021, 2nd edition 2024

To Helga

Preface

This book is intended for physics students (undergraduate/transition to graduate) who want to prepare for lectures on general relativity. Some knowledge of linear algebra and analysis are required. This textbook starts with basic topics such as vector space and vectors (chapter 1), dual space and covectors (chapter 2), tensors (chapter 3), etc.. Great importance is always attached to the clarity of the explanations and derivations of the topics. 31 Figures support these intentions.

General relativity is in its deeper sense a geometric theory. Therefore, the emphasis in this textbook has been placed on understanding space in its geometric configuration. Space is a component of the representation of the physical real. And it is thus the stage on which the physical processes and procedures show themselves.

Calculation tasks have been deliberately omitted. The focus is on understanding a topic. For this purpose, many examples and detailed extra introductions have been made. For exercises, there are enough examples in the relevant textbooks that can be used to deepen a topic. This book serves as a good basis for mastering tasks. If you have studied the book thoroughly, you will be prepared to start working on the physics of gravitation as described by general relativity.

Acknowledgments

Text and mathematical formulas were written with L^AT_EX. The Figures were created with the programme mathematica[®] from Wolfram Research, Inc. I would like to thank Simone Szurmant of ADDITIVE, Friedrichsdorf, Germany, and Thomas Ponweiser of uni software plus, Linz, Austria, for their support with problems and questions concerning mathematica[®].

Contents

1	Vector Space and Vectors	1
1.1	Vector Space	1
1.2	Basis	1
1.3	Components	3
1.4	Transformation	5
2	Dual Space and Covectors	11
2.1	Dual Space	11
2.2	Covectors (one-forms)	12
2.3	Transformation	13
2.4	Pictorial representation of covectors (one-forms)	14
3	Tensors	19
3.1	Definition and Notation	19
3.2	Tensor product	21
3.3	Tensor Transformation	23
4	Metric Tensors	25
4.1	Introduction	25
4.2	Inner product	25
4.3	Definition and components of \mathbf{g}	27
4.4	Signature of the metric	29
4.5	Integration	31
4.6	Tensor densities	33
4.7	Mapping of vectors into one-forms (covectors)	36
4.8	Dual metric tensor \mathbf{g}	36
4.9	Index raising and lowering	37
4.10	Magnitudes and dot products of one-forms	39
4.11	Compilation of the handling of tensors	40

5	Gradient	45
5.1	Differential operator	45
5.2	Covector field or gradient	46
6	Getting ready for curved space	51
6.1	Covariant derivative of a vector field	51
6.2	Covariant derivative and coordinate Transformation	59
6.3	Covariant derivative of a covector field	60
6.4	Connection and metric	62
6.5	Divergence and Laplacian	65
6.6	Local flatness theorem	69
7	Curved space	73
7.1	Manifold	73
7.2	Intrinsic vectors	75
7.3	Vector fields on manifolds	79
7.4	Riemann manifold	80
7.5	Parallel-transport of vectors	81
7.6	Geodesics	82
7.7	The curvature tensor	84
7.8	Symmetry properties of the Riemann tensor	89
7.9	Integral curves	93
7.10	Non-coordinate bases	96
7.11	Lie bracket	100
7.12	Torsion Tensor	103
7.13	The commutator of two covariant derivatives	104
7.14	Geodesic deviation	106
7.15	Bianchi identities	110
7.16	Ricci tensor	110
8	Differential forms	113
8.1	Antisymmetric tensors	113
8.2	Differential forms	115
8.3	Integration revisited	120
8.4	Oriented vector space	124
8.5	p -vectors	125
8.6	Dual spaces of p -forms and p -vectors	127
8.7	Hodge dual operator $*$	128
8.8	Dual map	134
8.9	Exterior derivative	137
8.10	Gradient, Curl and Divergence	142

8.11	General Stoke's Theorem	144
9	Introduction to Special Relativity (SR)	147
9.1	Inertial frame	147
9.2	Minkowski space (MS)	148
9.3	Lorentz transformation (LT)	155
9.4	Spacetime diagram	162
9.5	Time dilation	167
9.6	Length contraction	168
9.7	Vectors	170
9.8	Momentum and Energy	175
A	Some mathematical terms	183
B	Determinants and Square Matrices	185
B.1	Determinants	185
B.2	Useful formulas with matrix operations	187
C	Generalized Kronecker delta	191
C.1	Kronecker delta	191
C.2	Generalized Kronecker delta	192
D	Levi-Civita symbol	197
D.1	Vector cross and triple product	198

List of Figures

1.1	Position vector \vec{R} in two different coordinate systems.	3
1.2	Vector field $\vec{v} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$	4
2.1	Picture of one-form $\tilde{p} = \tilde{e}^x$	15
2.2	Picture of one-form $\tilde{p} = 4\tilde{e}^x$	15
2.3	The one-form \tilde{p} processes the vector \vec{v} to the number 2.	16
2.4	Graphic solution of $\tilde{p}(\vec{v}) = (\tilde{e}^x + 5\tilde{e}^y)(2\vec{e}_x + \vec{e}_y)$	17
3.1	No matter in which coordinate system a tensor \mathbf{T} is located, its properties remain unaffected.	20
4.1	A symbol of the interweaving of vector space \mathcal{V} and dual space \mathcal{V}^* with the metric tensor \mathbf{g} and \mathbf{g}	26
5.1	Pictorial difference between dx and $\tilde{d}(x)$	45
5.2	Scalar field and associated covector (one-form) gradient.	47
6.1	Transport of a vector field \vec{v} along a vector field \vec{u}	53
7.1	A chart ϕ maps an open set U of the manifold M	74
7.2	The overlap area on M is mapped to the grey coloured areas.	74
7.3	The point λ is mapped to the point \mathcal{P} in M	76
7.4	x^1 - x^2 coordinate grid.	85
7.5	Vector field $\vec{v} = 4\vec{e}_x + x\vec{e}_y$ and flow curve through point $\mathcal{P}(0, 1)$	94
7.6	Integral curves of $\vec{v} = -y\vec{e}_x + x\vec{e}_y$	95
7.7	Simplified geometric interpretation of the Lie bracket, as a difference vector of different flow curves.	100
7.8	Two straight lines $\mathbf{g}_1(\rho)$ and $\mathbf{g}_2(\rho)$ in Euclidean space with their separation vector \vec{s}	106
7.9	A set of initially parallel geodesics with tangent vector \vec{u} and separation vector \vec{s}	107

8.1	The four vector spaces $\wedge^p \mathcal{V}^*$, $\wedge^{n-p} \mathcal{V}^*$, $\wedge^p \mathcal{V}$ and $\wedge^{n-p} \mathcal{V}$ at point \mathcal{P} on a manifold M	127
9.1	Invariant interval hyperbolas for the distances d_1 (timelike) and d_2 (spacelike). The dashed straight lines, asymptotes of the hyperbolas, denote the lightlike events.	152
9.2	Hyperbola d_1 from Figure 9.1 in an alternative representation of clocks attached to positions.	153
9.3	Pictorial representation of constant intervals of magnitude 1 for the Euclidean space and the Minkowski space of special relativity..	154
9.4	\mathcal{K} spacetime diagram: The (t', x') coordinates of \mathcal{K}' related to the (t, x) coordinates of \mathcal{K} in the case of a x^+ -boost Lorentz transformation.	163
9.5	\mathcal{K}' spacetime diagram: The (t, x) coordinates of \mathcal{K} related to the (t', x') coordinates of \mathcal{K}' in the case of a x^+ -boost Lorentz transformation.	164
9.6	Event \mathcal{A} with its coordinates in \mathcal{K} and \mathcal{K}'	165
9.7	Spacetime hyperbola in the \mathcal{K} spacetime diagram ($v = \tanh \phi$).	166
9.8	Time calibration between the inertial frames \mathcal{K} and \mathcal{K}' ($v = \tanh \phi$).	167
9.9	Length comparison of a rod $l_{\mathcal{AB}}$ resting in \mathcal{K}' to its measured length $l_{\mathcal{AC}}$ in \mathcal{K}	169
9.10	CRF basis vectors with the corresponding 4-velocity at two events \mathcal{A} and \mathcal{B} of the world line.	176

Chapter 1

Vector Space and Vectors

1.1 Vector Space

A *vector space* \mathcal{V} contains the following arithmetic operations over its elements: addition and scaling. For our geometric investigations we call its elements *vector*. Vectors are drawn as arrows with value (length) and direction to make it geometrically visual, Figure 1.1 and Figure 1.2. There are also vector spaces over polynomials, matrices, tensors etc.

Let \vec{u}, \vec{v} and \vec{w} be vectors in \mathcal{V} , a and b real numbers, then the following *axioms* apply to a real vector space \mathcal{V} :

1. $\vec{v} + \vec{w} = \vec{w} + \vec{v} \in \mathcal{V}$,
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \in \mathcal{V}$,
3. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \in \mathcal{V}$,
4. $(a + b)\vec{v} = a\vec{v} + b\vec{v} \in \mathcal{V}$,
5. $(ab)\vec{v} = a(b\vec{v}) \in \mathcal{V}$,
6. $1 \cdot \vec{v} = \vec{v} \in \mathcal{V}$.

The identity element of \mathcal{V} is the *zero vector* $\vec{0}$: $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$.

1.2 Basis

In order to be able to operate calculus in a vector space, one introduces basis vectors which span the entire vector space \mathcal{V} . The number of necessary basis vectors corresponds to the dimension n of the vector space. The algebraic

relationship for an ordered set of basis vectors $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ and a set $\{a_i\}$ of real numbers is:

$$a^1\vec{e}_1 + a^2\vec{e}_2 + \dots + a^n\vec{e}_n = \vec{0}, \text{ only for } a^i = 0, \forall i. \quad (1.1)$$

This equation has *no* non-trivial solution. Basis vectors are thus *linearly independent*. No basis vector is a linear combination of other basis vectors.

Remark 1.1

You can of course define different bases for an n -dimensional vector space. Example: Cartesian, polar, etc.



Each vector \vec{v} is *uniquely determined* by the coefficients a^i using the basis vectors. These coefficients are called the components of the vector.

$$\vec{v} = a^1\vec{e}_1 + a^2\vec{e}_2 + \dots + a^n\vec{e}_n.$$

Proof:

We assume that there is a second representation of vector \vec{v} :

$$\vec{v} = b^1\vec{e}_1 + b^2\vec{e}_2 + \dots + b^n\vec{e}_n.$$

We form the difference of the vector to itself, obtaining the zero vector

$$\begin{aligned} \vec{0} &= \vec{v} - \vec{v} \\ &= (a^1\vec{e}_1 + a^2\vec{e}_2 + \dots + a^n\vec{e}_n) - (b^1\vec{e}_1 + b^2\vec{e}_2 + \dots + b^n\vec{e}_n) \\ &= (a^1 - b^1)\vec{e}_1 + (a^2 - b^2)\vec{e}_2 + \dots + (a^n - b^n)\vec{e}_n, \end{aligned}$$

and because of the linear independence of the basis vectors \vec{e}_i , equation (1.1) applies: $(a^i - b^i) = 0$.



1.3 Components

A vector \vec{v} in a 2-dimensional vector space \mathcal{V} with basis vectors \vec{e}_i can be described algebraically as follows:

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2. \quad (1.2)$$

The numbers v^i are called the *components* of the vector \vec{v} . In linear algebra, they are summarized in a *column vector*:

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \quad (1.3)$$

Remark 1.2

As stated in Section 1.1 in this textbook we work only with vector spaces based on the real numbers \mathbb{R} (components, scalars, functions, etc.).

■

A position vector \vec{R} is a directional vector from the origin of the coordinate system to a selected point of the vector space. The Figure 1.1 shows an example of a position vector \vec{R} in a 2-dimensional coordinate system. Once in a Cartesian coordinate system and the other in a general coordinate system. In contrast to a position vector $\vec{R} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$ (Figure 1.1), the vector

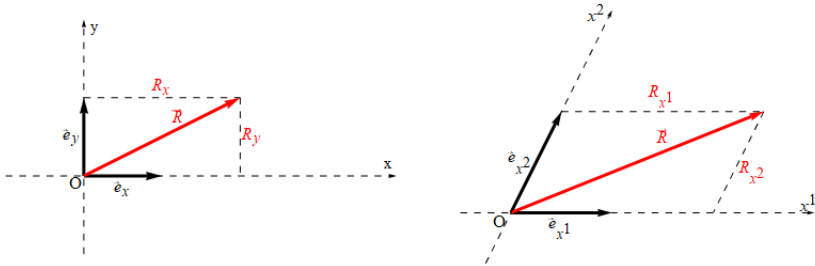


Figure 1.1: Position vector \vec{R} in two different coordinate systems.

$\vec{v} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$ represents a *vector field*. See Figure 1.2.

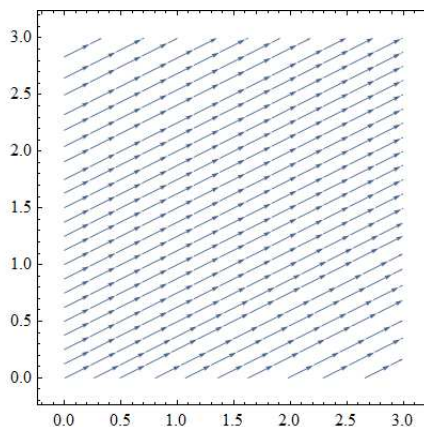


Figure 1.2: Vector field $\vec{v} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$.

Remark 1.3

The *components* of a vector \vec{v} of a vector space \mathcal{V} are always provided with a *superscript* (upstairs, upper) index, the *basis vectors* \vec{e}_i of a vector space \mathcal{V} always with a subscript (downstairs, lower) index. In the dual space \mathcal{V}^* (see Chapter 2), the situation is reversed.

■

Box 1.1

We mostly use the *Einstein summation convention*:

$$\sum_{i=1}^n v^i \vec{e}_i = v^i \vec{e}_i, \quad \sum_k a^k b_k, \text{ and so on.} \quad (1.4)$$

In other words, identical superscripts and subscripts are added up. This is the so called *contraction*. See later sections.

Of course, the summation indices can be supplemented by other symbols (*dummy indices*). But attention: already used letters, symbols in the term to be edited may **not** be reused!

$$a^k b_k = a^i b_i = a^{\spadesuit} b_{\spadesuit} = \dots,$$

but: $a^k b_k c^i \neq a^i b_i c^i$

Not summed up indices are called *free indices*. ■

1.4 Transformation

In this text, mainly the *coordinate basis* is used, which is expressed by means of the partial differential operator, applied to the coordinate system $\{x^i\}$:

$$\vec{e}_k \equiv \frac{\partial}{\partial x^k} \equiv \partial_k. \quad (1.5)$$

Partial derivatives and coordinate basis vectors as defined by equation (1.5) are the same because $\frac{\partial}{\partial x^k}$ follows the x^k coordinate line. An important property of a coordinate basis is that its commutator vanishes (partial derivatives commute).

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = [\partial_i, \partial_j] = 0. \text{ No restrictions for } i \text{ and } j. \quad (1.6)$$

Remark 1.4

It should be noted that in general the basis vectors \vec{e}_k of a coordinate basis, equation (1.5), are *not* normalized.



Box 1.2

An essential feature of a coordinate system is that coordinate x^i is constant along coordinate $x^j, i \neq j$. Thus the relation

$$\frac{\partial x^i}{\partial x^j} = \delta_j^i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

applies, which we will repeatedly encounter in this textbook.

The *Kronecker delta* is written $\delta_{ij}, \delta_j^i, \delta^{ij}$ and is defined by

$$\delta_{ij}, \delta_j^i, \delta^{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \quad (1.7)$$

As an example we treat the transition from Cartesian coordinates to polar coordinates and vice versa. It is known:

$$x = r \cos \theta, y = r \sin \theta. \quad (1.8)$$

And with the partial derivative chain rule we get the basis vectors for polar coordinates:

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r} \frac{\partial}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial}{\partial y} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \\ \text{or } \vec{e}_r &= \cos \theta \vec{e}_x + \sin \theta \vec{e}_y. \end{aligned} \quad (1.9)$$

And,

$$\begin{aligned} \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial}{\partial y} = -r \sin \theta \frac{\partial}{\partial x} + r \cos \theta \frac{\partial}{\partial y}, \\ \text{or } \vec{e}_\theta &= -r \sin \theta \vec{e}_x + r \cos \theta \vec{e}_y. \end{aligned} \quad (1.10)$$

We summarize this compactly into a *matrix representation*:

$$\begin{pmatrix} \vec{e}_r & \vec{e}_\theta \end{pmatrix} = \begin{pmatrix} \vec{e}_x & \vec{e}_y \end{pmatrix} \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}. \quad (1.11)$$

Note that subscript entries are grouped together as *row vectors*.

The coordinate basis transformation from the coordinate system $\{x, y\}$ to the coordinate system $\{x', y'\}$ is

$$\begin{pmatrix} \partial_{x'} & \partial_{y'} \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix}. \quad (1.12)$$

And in general terms, a coordinate basis transformation, *transformation matrix* F , from unprimed coordinates to primed coordinates is

$$\vec{e}_{j'} = \vec{e}_i F_{j'}^i, \quad F_{j'}^i = \frac{\partial x^i}{\partial x^{j'}}; \quad 2\text{-dim: } F_{j'}^i = \begin{pmatrix} \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y'} \\ \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y'} \end{pmatrix}. \quad (1.13)$$

And what does the reverse transformation look like from primed coordinates to unprimed coordinates? With equation (1.13) we can write:

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} = \begin{pmatrix} \partial_{x'} & \partial_{y'} \end{pmatrix} \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}. \quad (1.14)$$

or

$$\vec{e}_i = \vec{e}_{k'} R_i^{k'}, \quad R_i^{k'} = \frac{\partial x^{k'}}{\partial x^i}; \quad 2\text{-dim: } R_i^{k'} = \begin{pmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} \end{pmatrix}. \quad (1.15)$$