Introduction to the Geometrical Foundations of General Relativity

A Compendium of Tensor Calculus for Physicists

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To Helga

Preface

This book is intended for physics students (undergraduate/transition to graduate) who want to prepare for lectures on general relativity. Some knowledge of linear algebra and analysis are required. This textbook starts with basic topics such as vector space and vectors (chapter 1), dual space and covectors (chapter 2), tensors (chapter 3), etc.. Great importance is always attached to the clarity of the explanations and derivations of the topics. 31 Figures support these intentions.

General relativity is in its deeper sense a geometric theory. Therefore, the emphasis in this textbook has been placed on understanding space in its geometric configuration. Space is a component of the representation of the physical real. And it is thus the stage on which the physical processes and procedures show themselves.

Calculation tasks have been deliberately omitted. The focus is on understanding a topic. For this purpose, many examples and detailed extra introductions have been made. For exercises, there are enough examples in the relevant textbooks that can be used to deepen a topic. This book serves as a good basis for mastering tasks. If you have studied the book thoroughly, you will be prepared to start working on the physics of gravitation as described by general relativity.

Acknowledgments

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Chapter 1

Vector Space and Vectors

1.1 Vector Space

A *vector space* $\mathcal{V}$ contains the following arithmetic operations over its elements: addition and scaling. For our geometric investigations we call its elements *vector*. Vectors are drwan as arrows with value (length) and direction to make it geometrically visual, Figure 1.1 and Figure 1.2. There are also vector spaces over polynomials, matrices, tensors etc.

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors in $\mathcal{V}$, a and b real numbers, then the following *axioms* apply to a real vector space $\mathcal{V}$:

1. $\vec{v} + \vec{w} = \vec{w} + \vec{v} \in \mathcal{V}$,
2. $\vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \in \mathcal{V}$,
3. $a(\vec{v} + \vec{w}) = a\vec{v} + a\vec{w} \in \mathcal{V}$,
4. $(a + b)\vec{v} = a\vec{v} + b\vec{v} \in \mathcal{V}$,
5. $(ab)\vec{v} = a(b\vec{v}) \in \mathcal{V}$,
6. $1 \cdot \vec{v} = \vec{v} \in \mathcal{V}$.

The identity element of $\mathcal{V}$ is the *zero vector* $\vec{0}$: $\vec{v} + \vec{0} = \vec{0} + \vec{v} = \vec{v}$.

1.2 Basis

In order to be able to operate calculus in a vector space, one introduces basis vectors which span the entire vector space $\mathcal{V}$. The number of necessary basis vectors corresponds to the dimension n of the vector space. The algebraic relationship for an ordered set of basis vectors $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n\}$ and a set $\{a_i\}$ of real numbers is:

$$a^1\vec{e}_1 + a^2\vec{e}_2 + \cdots + a^n\vec{e}_n = \vec{0}, \text{ only for } a^i = 0, \forall i. \tag{1.1}$$

This equation has *no* non-trivial solution. Basis vectors are thus *linearly independent* . No basis vector is a linear combination of other basis vectors.

Remark 1.1
You can of course define different bases for an n-dimensional vector space. Example: Cartesian, polar, etc.

■

Each vector $\vec{v}$ is *uniquely determined* by the coefficients a^i using the basis vectors. These coefficients are called the components of the vector.

$$\vec{v} = a^1\vec{e}_1 + a^2\vec{e}_2 + \cdots + a^n\vec{e}_n.$$

Proof:
We assume that there is a second representation of vector $\vec{v}$:

$$\vec{v} = b^1\vec{e}_1 + b^2\vec{e}_2 + \cdots + b^n\vec{e}_n.$$

We form the difference of the vector to itself, obtaining the zero vector

$$\begin{aligned}\vec{0} &= \vec{v} - \vec{v} \\ &= (a^1\vec{e}_1 + a^2\vec{e}_2 + \cdots + a^n\vec{e}_n) - (b^1\vec{e}_1 + b^2\vec{e}_2 + \cdots + b^n\vec{e}_n) \\ &= (a^1 - b^1)\vec{e}_1 + (a^2 - b^2)\vec{e}_2 + \cdots + (a^n - b^n)\vec{e}_n,\end{aligned}$$

and because of the linear independence of the basis vectors $\vec{e}_i$, equation (1.1) applies: $(a^i - b^i) = 0$.

■

1.3 Components

A vector $\vec{v}$ in a 2-dimensional vector space $\mathcal{V}$ with basis vectors $\vec{e}_i$ can be described algebraically as follows:

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2. \tag{1.2}$$

The numbers v^i are called the *components* of the vector $\vec{v}$. In linear algebra, they are summarized in a *column vector*:

$$\vec{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} \tag{1.3}$$

Remark 1.2
As stated in Section 1.1 in this textbook we work only with vector spaces based on the real numbers $\mathbb{R}$ (components, scalars, functions, etc.).

■

A position vector $\vec{R}$ is a directional vector from the origin of the coordinate system to a selected point of the vector space. The Figure 1.1 shows an example of a position vector $\vec{R}$ in a 2-dimensional coordinate system. Once in a Cartesian coordinate system and the other in a general coordinate system. In contrast to a position vector $\vec{R} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$ (Figure 1.1), the vector

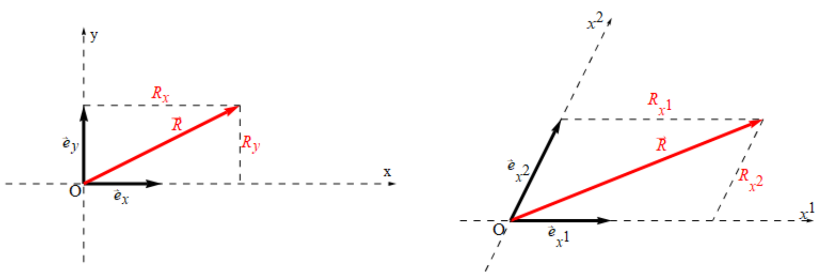

Figure 1.1: Position vector $\vec{R}$ in two different coordinate systems.

$\vec{v} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$ represents a *vector field*. See Figure 1.2.

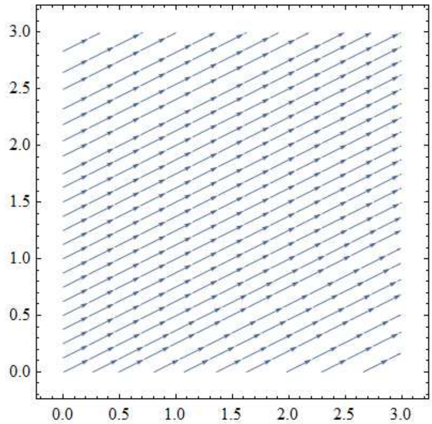

Figure 1.2: Vector field $\vec{v} = 2 \cdot \vec{e}_1 + 1 \cdot \vec{e}_2$.

Remark 1.3
The *components* of a vector $\vec{v}$ of a vector space $\mathcal{V}$ are always provided with a *superscript* (upstairs, upper) index, the *basis vectors* $\vec{e}_i$ of a vector space $\mathcal{V}$ always with a subscript (downstairs, lower) index. In the dual space $\mathcal{V}^\star$ (see Chapter 2), the situation is reversed.

■

Box 1.1
We mostly use the *Einstein summation convention*:

$$\sum_{i=1}^{n} v^i \vec{e}_i = v^i \vec{e}_i, \quad \sum_k a^k b_k, \text{and so on.} \tag{1.4}$$

In other words, identical superscripts and subscripts are added up. This is the so called *contraction* . See later sections.
Of course, the summation indices can be supplemented by other symbols (*dummy indices*) . But attention: already used letters, symbols in the term to be edited may **not** be reused!

$$a^k b_k = a^i b_i = a^\spadesuit b_\spadesuit = \cdots,$$
$$\text{but:} \quad a^k b_k c^i \neq a^i b_i c^i$$

Not summed up indices are called *free* indices . ■

1.4 Transformation

In this text, mainly the *coordinate basis* is used, which is expressed by means of the partial differential operator, applied to the coordinate system $\{x^i\}$:

$$\vec{e}_k \equiv \frac{\partial}{\partial x^k} \equiv \partial_k. \tag{1.5}$$

Partial derivatives and coordinate basis vectors as defined by equation (1.5) are the same because $\frac{\partial}{\partial x^k}$ follows the x^k coordinate line. An important property of a coordinate basis is that its commutator vanishes (partial derivatives commutate).

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = [\partial_i, \partial_j] = 0. \text{ No restrictions for } i \text{ and } j. \tag{1.6}$$

Remark 1.4
It should be noted that in general the basis vectors $\vec{e}_k$ of a coordinate basis, equation (1.5), are *not* normalized.

■

Box 1.2
An essential feature of a coordinate system is that coordinate x^i is constant along coordinate $x^j, i \neq j$. Thus the relation

$$\frac{\partial x^i}{\partial x^j} = \delta^i_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

applies, which we will repeatedly encounter in this textbook.
The *Kronecker delta* is written $\delta_{ij}, \delta^i_j, \delta^{ij}$ and is defined by

$$\delta_{ij}, \delta^i_j, \delta^{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{1.7}$$

As an example we treat the transition from Cartesian coordinates to polar coordinates and vice versa. It is known:

$$x = r\cos\theta, y = r\sin\theta. \tag{1.8}$$

And with the partial derivative chain rule we get the basis vectors for polar coordinates:

$$\begin{aligned} \frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \\ \text{or} \quad \vec{e}_r &= \cos\theta \;\; \vec{e}_x + \sin\theta\,\vec{e}_y. \end{aligned} \tag{1.9}$$

And,

$$\begin{aligned} \frac{\partial}{\partial \theta} &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}, \\ \text{or} \quad \vec{e}_\theta &= -r\sin\theta \;\; \vec{e}_x + r\cos\theta\,\vec{e}_y. \end{aligned} \tag{1.10}$$

We summarize this compactly into a *matrix representation*:

$$\begin{pmatrix} \vec{e}_r & \vec{e}_\theta \end{pmatrix} = \begin{pmatrix} \vec{e}_x & \vec{e}_y \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}. \tag{1.11}$$

Note that subscript entries are grouped together as *row vectors.*

The coordinate basis transformation from the coordinate system $\{x, y\}$ to the coordinate system $\{x^{'}, y^{'}\}$ is

$$\begin{pmatrix} \partial_{x^{'}} & \partial_{y^{'}} \end{pmatrix} = \begin{pmatrix} \partial_x & \partial_y \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial x^{'}} & \frac{\partial x}{\partial y^{'}} \\ \frac{\partial y}{\partial x^{'}} & \frac{\partial y}{\partial y^{'}} \end{pmatrix}. \tag{1.12}$$

And in general terms, a coordinate basis transformation, *transformation matrix* F, from unprimed coordinates to primed coordinates is

$$\vec{e}_{j^{'}} = \vec{e}_i F^i_{j^{'}}, \quad F^i_{j^{'}} = \frac{\partial x^i}{\partial x^{j^{'}}}; \quad \text{2-dim: } F^i_{j^{'}} = \begin{pmatrix} \frac{\partial x}{\partial x^{'}} & \frac{\partial x}{\partial y^{'}} \\ \frac{\partial y}{\partial x^{'}} & \frac{\partial y}{\partial y^{'}} \end{pmatrix}. \tag{1.13}$$

And what does the reverse transformation look like from primed coordinates to unprimed coordinates? With equation (1.13) we can write:

$$\begin{pmatrix} \partial_x & \partial_y \end{pmatrix} = \begin{pmatrix} \partial_{x^{'}} & \partial_{y^{'}} \end{pmatrix} \begin{pmatrix} \frac{\partial x^{'}}{\partial x} & \frac{\partial x^{'}}{\partial y} \\ \frac{\partial y^{'}}{\partial x} & \frac{\partial y^{'}}{\partial y} \end{pmatrix}. \tag{1.14}$$

or

$$\vec{e}_i = \vec{e}_{k^{'}} R^{k^{'}}_i, \quad R^{k^{'}}_i = \frac{\partial x^{k^{'}}}{\partial x^i}; \quad \text{2-dim: } R^{k^{'}}_i = \begin{pmatrix} \frac{\partial x^{'}}{\partial x} & \frac{\partial x^{'}}{\partial y} \\ \frac{\partial y^{'}}{\partial x} & \frac{\partial y^{'}}{\partial y} \end{pmatrix}. \tag{1.15}$$