Mathematical Methods of Lagrangian and Hamiltonian Mechanics

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To Helga

Preface

This book is intended to help students of physics and other branches of science in the first semesters of their studies to better understand the applied mathematical methods of Lagrangian and Hamiltonian mechanics. The book has the benefit of learning, in addition to the physical processes of classical mechanics, with focus on Lagrangian and Hamiltonian mechanics, the mathematical methods that are equally needed in other branches of physics. These include: Vector calculus, matrix calculus, tensor calculus, differential equations, derivative chain rule, Taylor series, differential geometry, implicit function theorem, coordinate transformation (Jacobian), curvilinear coordinates, Legendre transformation, and much more.

Chapter 1 describes the basics of Newtonian mechanics in a review. In addition to Newton's laws, the two-body problem is dealt with in detail. Kepler's laws are a by-product of this.

Chapter 2 explains the origins of the variation technique with its historical origin in the brachistochrone problem. After introducing generalised coordinates and applying Newton's principle of determinacy, the Lagrangian approach for mechanical systems is derived. The conservation laws play an important role in this context. Applications are shown for motions in a central field. The Lagrangian dynamics for oscillations with the various modes is discussed in depth. The application of linear algebra (eigenvectors, normal coordinates) is treated in great detail.

Chapter 3 develops the Hamiltonian dynamics for mechanical systems. The transition from the configuration space of Lagrangian mechanics to the symplectic phase space of Hamiltonian mechanics (Legendre transformation) is discussed. An additional section deals with Routh's procedure, which can be described as a mixture of Lagrangian and Hamiltonian mechanics.

The extension of the permissible transformations of the variables (q^i, p_i) of Hamiltonian mechanics in comparison to Lagrangian approach leads us to the canonical transformations, Chapter 4. Here the generating functions of the canonical transformations are derived with the help of the Legendre transformation. The symplectic relationship of canonical transformations is clearly worked out.

In Chapter 5, the Hamiltonian equations of motion are described using the Poisson formalism, which provides the equations of motion with a symmetrical form. Further topics such as constants of motion, Jacobi identity, canonical invariance, Liouville's theorem, etc. are treated in detail.

Hamilton-Jacobi theory, Chapter 6, considers the interesting approach of finding a canonical transformation in which the phase space coordinates and the new Hamiltonian are all constant. This is discussed in depth and the student is given a procedure for solving a mechanical system.

A canonical transformation, the so-called action-angle variable, which is discussed in Chapter 7, is suitable for periodic phase orbits. The important field of adiabatic invariants with reference to quantum mechanics is also discussed.

The texts are supported with many graphics and help the student to grasp the current topic more intuitively. All chapters contain many exercises. The student is encouraged to first try to solve the exercises independently before consulting the solutions provided.

Bernd Wichmann
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Chapter 1

Basics

1.1 Gravitation

Gravitation is one of the four basic forces, along with electromagnetism, strong nuclear force and weak nuclear force. Until today we don't know *what* makes the planets revolve around the sun, but only *how* they revolve. This how is united in the famous Newtonian[1] gravitation formula

$$F = G\frac{m\,M}{r^2}.$$

The gravitational force between two masses[2] is always *attractive*, never repulsive. Isaac Newton derived the law of gravitation from the three Kepler's[3] laws (see Box 1.1). The gravitation formula describes that two masses attract each other with a force proportional to the masses of the objects and inversely proportional to the square of their distances. Established for the planetary motion around the sun, however, it has universal validity: *Each object attracts all the others.* The gravitational force has its validity in the whole universe observable until today. Even among galaxies, which are tens of millions of light years apart, it still seems to work. Although the gravitational force is far weaker than the electric force by a factor of 4.17×10^{42} (comparison between the electric repulsion of two electrons and their attraction by gravitation), it creates planets, stars and galaxies with its almost infinite range.

[1]Isaac Newton (1643-1727), English mathematician and physicist

[2]Each mass distribution can be assigned to a center of mass, at which the forces, here the gravitational force, act, just as if the entire mass were concentrated at this point. r is the distance between the two centers of mass of m and M.

[3]Johannes Kepler (1571-1630), German mathematician and astronomer

Remark 1.1

In this chapter, the masses should be given an index G for *gravity*. However, we will omit this for the sake of clarity. The gravitational mass is responsible for the gravitational force. In Section 1.8 where we deal with Newton's laws, the mass is given the index I to clarify the *principle of inertia*, the mass resists a change in its position. However, since the two masses do not differ in their scalar value $m_G = m_I$ according to the present state of knowledge, we omit the indices in the further course of the text.

■

Box 1.1

Kepler's laws

The astronomer Tycho Brahe[a] of the 16th century observed the course of the planets. He recorded with high accuracy their position in the night sky and compiled the data in extensive tables. The astronomer and mathematician Johannes Kepler sifted and analysed the data Brahe had collected. He summarized the result in three laws .

1. Each planet moves around the sun on an ellipse, in one focus of which is the sun.
2. The radius vector from the sun to the planet passes over equal areas in equal time intervals.
3. The squares of the orbital periods of any two planets behave like the cubes of the semimajor axes of their orbits:

$$\frac{{T_1}^2}{{T_2}^2} = \frac{{a_1}^3}{{a_2}^3}.$$

[a]Tycho Brahe (1546-1601), Danish astronomer

1.2 Scalar field Φ

We imagine an empty universe. Now we insert a point-like object with the property *mass* and his magnitude M into this empty space at an arbitrary position. This mass creates a *number field* Φ, around itself. This scalar field is also called *potential field*, or just *potential* for short. How will these numbers arrange themselves? Since we assume an isotropic space which contains no preferred direction, this scalar field has a spherical symmetry. Since we want to develop gravitational forces from this number field, we introduce a minus sign and the gravitational constant G. Let this scalar field now be described mathematically as follows

$$\Phi(r) = -G\,\frac{M}{r}, \tag{1.1}$$

where r is the distance of any point of the space to the mass M.

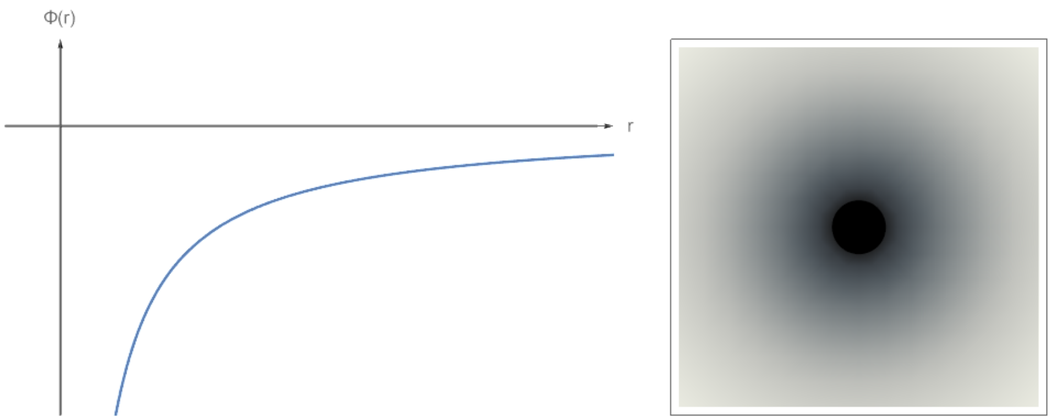

Figure 1.1 The scalar field $\Phi(r)$ drawn as a function (left), and as a density plot (right).

Figure 1.1 shows us graphically the behaviour of this central scalar field[4] of equation (1.1). Once plotted as a function, and secondly as a colouring of the numbers. The blue area corresponds to high negative numbers. In the red area

[4] A scalar field $f(P)$ is called a central field if there is a point O where $f(P)$ depends only on the distance r between the point P and O, i.e. $f(P) = \Phi(r)$.

the numbers approach zero.

The best way to analyse a scalar field is to form the *gradient* of the number field. A gradient shows the *contour lines*, lines of constant value, and the *direction* in which the number field takes on higher values. The gradient of a scalar field can be compared to a topographical map with its contours of equal elevation. The narrower the contour lines, the steeper the terrain. And the greater the magnitude of the gradient. The gradient of a scalar field is a *one-form*, a *covector* (see [Wic21]). Applied to (1.1) we get

$$\operatorname{grad}\Phi = \tilde{\mathrm{d}}\,\Phi = \partial_i \Phi\, \tilde{e}^i, \tag{1.2}$$

with the covariant components $\partial_i \Phi$, represented as stacked contour lines, surfaces etc. and the dual space basis vectors $\tilde{e}^i$.

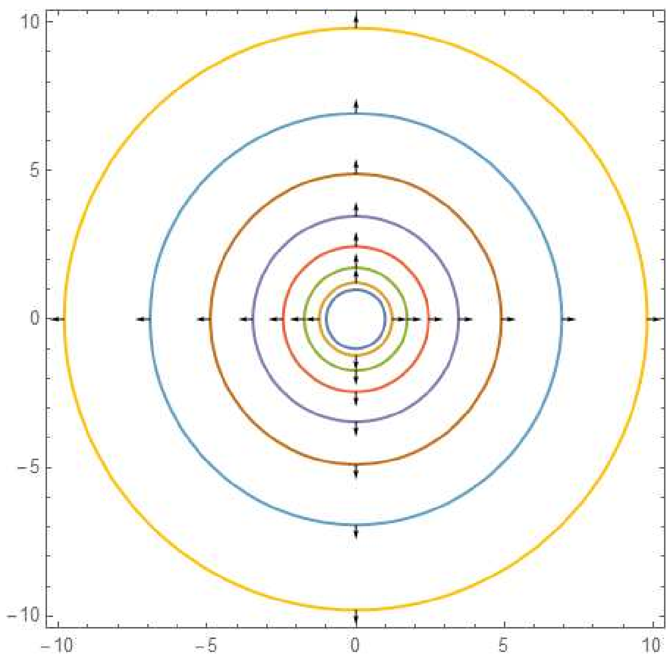

Figure 1.2 Contour lines and direction arrows of increasing values of grad Φ.

Figure 1.2 shows the gradient field $\tilde{\mathrm{d}}\,\Phi$ of the scalar field Φ of equation (1.1). It displays the contour lines of the same value and the directional arrows to indicate ascending values. The higher magnitudes of the gradient field $\tilde{\mathrm{d}}\,\Phi$ can be recognised by the denser contour lines as r becomes smaller.

We want to transform the covariant components $\partial_i \Phi$ into contravariant vector components by means of the metric $\mathbf{g}$ and index raising. Due to the spherical symmetric configuration of $\Phi(r)$ we use spherical coordinates (r, θ, ϕ).

$$\partial^j \Phi = \partial_i \Phi \, g^{ij}, \tag{1.3}$$

with ∂^j as a symbol for a partial derivative of a scalar leading to a contravariant component of a vector. We use ∂^j only when it needs to be emphasised in the context for distinction. It should be noted that a gradient of a scalar field f, $\operatorname{grad} f$, since it is a one-form, is always to be calculated with $\partial_i f$, and subsequently, if a vector notation of the gradient field ∇f is necessary, to be transferred into contravariant vector components with index raising. The following applies to the scalar field $\Phi(r)$ under consideration, (1.1):

$$\begin{aligned}
\nabla\,\Phi(r) &= \partial^j \Phi(r) \hat{e}_j = \partial_i \Phi(r)\, g^{ij}\, \hat{e}_j \\
&= \partial_r \Phi(r)\, g^{rr}\, \hat{e}_r + \partial_\theta \Phi(r)\, g^{\theta\theta}\, \hat{e}_\theta + \partial_\phi \Phi(r)\, g^{\phi\phi}\, \hat{e}_\phi \\
&= \partial_r \Phi(r) \cdot 1 \cdot \hat{e}_r + 0 \cdot \frac{1}{r} \cdot \hat{e}_\theta + 0 \cdot \frac{1}{r \sin\theta} \cdot \hat{e}_\phi \\
&= \partial_r \Phi(r)\, \hat{e}_r \\
&= G \frac{M}{r^2}\, \hat{e}_r .
\end{aligned} \tag{1.4}$$

The gravitational field $\vec{g}$ of a mass M is a central vector field[5] and is written as

$$\boxed{\vec{g} = -\nabla \Phi(r) = -G \frac{M}{r^2}\, \hat{e}_r,} \tag{1.5}$$

[5]A vector field $\vec{g}(P)$ is called a central vector field if there is a point O such that all vectors $\vec{g}(P)$ lie on straight lines passing through O and their magnitude depends only on the distance r between the point P and O, i.e. $\vec{g}(P) = f(r)\hat{e}_r$, where $\hat{e}_r$ is the unit vector of the straight line.

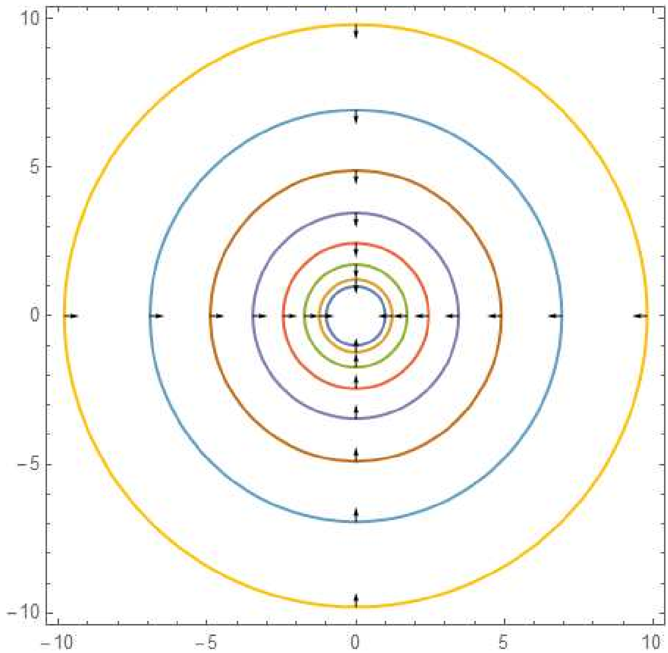

Figure 1.3 Gravitational field of a mass M with the contour lines of equal field strength, equation (1.5).

and shown again in Figure 1.3 with its contour lines of equal strength and the vectors $\hat{e}_r$ directed towards the center of mass.