1

Heat Transfer

1.1 In General

1.1.1 Heat

1.1.1.1 What?

A first description of what heat is comes from thermodynamics, a discipline that considers everything as a system surrounded by an environment, with which energy is exchanged. Can be the system: a material, a building assembly, a building, a heating, ventilating, air Conditioning (HVAC) installation, even a whole city. Energy transmitted as work between both looks purposeful and organized, energy transmitted as heat diffuse and chaotic. A second description of what heat is resides in particle physics, where heat refers to the statistically distributed kinetic energy of atoms and free electrons. Whatever, heat is the least noble, most diffuse form of energy, to which each nobler form degrades, see the second law of thermodynamics.

1.1.1.2 Sensible Heat

Sensible heat is unambiguously linked to what people feel, the temperature, see below. Transferring sensible heat, be it by conduction, convection and radiation, demands differences in temperature.

Conduction refers to the heat flow in a medium induced by its vibrating atoms, whose spheres of influence collide, and the movement of free electrons. Heat transmitted between solids at different temperature in ideal contact with each other and between points in these is straightforwardly conduction-based. This is also the case between gases and liquids and contacting surfaces. According to the second law of thermodynamics, conduction always moves direction lower temperatures. It needs a medium and does not induce macroscopic movement.

Convection in turn is the result of macroscopic movement in liquids and gases, in which temperature differences exist or that touch colder or warmer surfaces. As well external forces, differences in density or both together are inducing such movements and fix the type of convection generated: forced, natural or mixed. Convection needs a medium.

Radiation finally concerns heat transferred between surfaces due to them emitting and absorbing electromagnetic waves. Above 0 K, each surface radiates. When two

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or more surfaces at different temperature see each other, the result is heat exchanged between them. Radiation does not need a medium, while the laws governing it are very different from these shaping conduction and convection.

1.1.1.3 Latent Heat

Latent heat is linked to the changes of state between solid, liquid and gaseous. A release, for example liquid evaporating, or a deposit, for example liquid turning solid, require no differences in temperature, although both will impact the amounts of heat moving. Water evaporating absorbs the sensible heat of evaporation, so creates a heat sink. When then the water vapour so formed moves to a colder spot, where it condenses, the sensible heat of evaporation is emitted again, creating a heat source. Such sources and sinks not only impact the temperature profile in materials and assemblies, but they also have quite some impact on the sensible heat transferred.

1.1.2 Temperature

The temperature, described above as what people feel, mirrors the heat quality. Higher temperatures stay for more quality. Thanks to the related increase in kinetic energy of atoms and free electrons, the result is more exergy, which represents the potential to convert more heat into work via a cyclic process. Instead, lower temperatures and related decrease in the kinetic energy of atoms and free electrons stay for less exergy. Heightening the temperature of a system requires warming it, lowering the temperature of a system cooling it. Like any potential, temperature is a scalar. Sensing it is not a problem but measuring it is. Happily, many material properties depend on temperature, which allows its indirect quantification. A mercury thermometer allows scaling by using the volumetric expansion of mercury when heated and its volumetric contraction when cooled. In a Pt100 thermometer, the change of the electrical resistance of a platinum wire with temperature is used, while for thermocouples the varying contact potential between two metals is.

The SI system uses two temperature scales, one empiric, the degree Celsius (°C) with θ as symbol, and the other thermodynamic, the degree Kelvin (K) with T as symbol. 0 °C coincides with the triple point of water, and 100 °C with its boiling point at 1 Atmosphere. 0 K instead stands for the absolute zero, and 273.15 K for the triple point of water. Temperature differences are given in K, temperatures in °C or in K with as relation between both:

$$T = \theta + 273.15$$

Instead of degree Celsius (°C), the USA still uses degree Fahrenheit (°F). The link between the two is:

$$^{\circ}F = 32 + 9/5^{\circ}C$$

1.1.3 Why are Heat and Temperature so Compelling?

That heat is an issue in buildings, follows from the human demand for thermal comfort. In cold and temperate climates, the comfort temperatures required demand

heating during the colder months. In most cases, the heat sources still used are oil and natural gas. Their overall share in delivering the end energy needed causes such CO_2 release, that since the second half of the 1970s, energy efficiency became imperative. A prime opportunity to realize is by minimizing the heat loss through the building envelope. Knowing which envelope properties have a decisive influence on that, became a necessity for designers and builders.

Which temperatures are considered as being important depends on the situation and possible consequences. In winter, inside surface temperatures close to the air temperature indoors felt as comfortable upgrade thermal comfort. Instead, those much lower not only degrade the thermal comfort but also increase mould and surface condensation risk, both perceived as triggering healthiness. Also, too high summer temperatures indoors clash with thermal comfort and IAQ, while high temperature differences across outer assemblies increase the air and moisture movement in them, the thermal stresses experienced and crack risk. Large temperature gradients also favour the displacement of dissolved salts, while high temperatures accelerate the chemical breakdown of synthetics. Furthermore, too many temperature fluctuations from above to below freezing may damage wet, frost-sensitive porous materials. Whether all these effects will remain controllable, depends on how building assemblies are designed and built.

1.1.4 Some Definitions

Amount of heat, symbol Q, unit [J]

Quantifies the energy exchanged as heat. As heat is a scalar, the amount also is.

Heat flow, symbol Φ, unit [J/s] = [W]

Stands for the heat exchanged per unit of time. Heat flow is a measure for 'power', thus, a scalar.

Heat flux, symbol $\mathbf{q}$, unit [W/m^2]

Quantifies the heat exchanged per unit of time through a unit surface normal to the flow direction. The flux so is a vector with same direction as the surface vector. Its components in Cartesian coordinates are q_x, q_y, q_z, in polar coordinates q_R, q_ϕ, q_Θ.

Solving a heat transfer problem now means determining the scalar temperature field (T) and the vectorial heat fluxes field ($\mathbf{q}$). Computing both so requires a scalar and a vector equation.

1.2 Conduction

1.2.1 Conservation of Energy

A first relation between the heat flux ($\mathbf{q}$) and temperature (T) follows from the conservation of energy axiom. In case an infinitely small material volume is the system and what is around the environment, then, without mass displacement, the energy balance between both writes as:

$$\mathrm{d}\Phi + \mathrm{d}\Psi = \mathrm{d}U + \mathrm{d}W \tag{1.1}$$

with $\mathrm{d}\Phi$ the resulting heat flow between system and environment, $\mathrm{d}\Psi$ the heat dissipated uniformly in the system, $\mathrm{d}U$ the change in the system's internal energy and $\mathrm{d}W$ the labour exchanged with the environment, all per unit of time. Dissipation could include the heat produced by an exothermic reaction, the heat absorbed by an endothermic reaction, the Joule effect by an electric current passing through, latent heat released or absorbed, etc. The labour exchanged equals:

$$\mathrm{d}W = P\mathrm{d}\left(\mathrm{d}V\right) = P\,\mathrm{d}^2V$$

with P the pressure exerted in Pa. The conservation balance so states that the heat exchanged ($=\mathrm{d}\Phi$), released or absorbed, modifies the internal energy in this infinitely small material volume, while causing a labour exchange with the environment. If isobaric, the balance reshuffles to

$$\mathrm{d}(U + P\mathrm{d}V) = \mathrm{d}Q + \mathrm{d}E$$

with $U + P\mathrm{d}V$ the enthalpy (H). The resulting heat flow, the change in enthalpy and the heat dissipated now write as:

$$\mathrm{d}\Phi = -\mathrm{d}ix(\mathbf{q})\mathrm{d}V \quad \mathrm{d}H = \left|\frac{\partial(\rho c_\mathrm{p}T)}{\partial t}\right|\mathrm{d}V \quad \mathrm{d}\Psi = \Phi'\mathrm{d}V$$

with c_p the specific heat capacity at constant pressure of the material (J/(kg·K)), ρ its density (kg/m^3) and Φ' the heat dissipated per unit of time and volume, positive if a source, negative if a sink. The three turn the conservation equation into:

$$\left(\mathrm{div}(\mathbf{q}) + \Phi' + \frac{\partial(\rho c_\mathrm{p}T)}{\partial t}\right)\mathrm{d}V = 0 \tag{1.2}$$

For solids and liquids, the specific heat capacity hardly depends on the change of state. So, one value, symbol c, can be used with the product ρc equal to the volumetric specific heat capacity. For gases, the value varies with the change of state, giving as relation between the specific heat capacity at constant pressure (c_p) and at constant volume (c_v):

$$c_\mathrm{p} = c_\mathrm{v} + R$$

with R the specific gas constant (in Pa·m^3/(kg·K)). Because conservation of energy now holds for any infinitely small material volume, the relation between heat flux ($\mathbf{q}$) and temperature (T) so becomes:

$$\mathrm{div}\,\mathbf{q} = -\frac{\partial(\rho cT)}{\partial t} - \Phi' \tag{1.3}$$

1.2.2 Conduction Laws

1.2.2.1 First Law

The name 'first law' is given to the empirical vector equation between heat flux and temperature, advanced by the French physicist Fourier:

$$\mathbf{q} = -\lambda\,\mathbf{grad}\,T = -\lambda\,\mathbf{grad}\,\theta \tag{1.4}$$

It states that the conductive heat flux in a point somewhere in a solid, liquid or gas varies proportionally to the temperature gradient there. The proportionality value λ figures as a material property called the 'thermal conductivity' with as units W/(m·K). It expresses the ability of a medium to conduct heat. The minus sign indicates that the flux and the temperature gradient, which as vector goes from colder to warmer, oppose each other. Thermodynamics in fact learns that, if not forced externally, heat always moves direction colder. Otherwise, the entropy would decrease without energy input, which is impossible. Following observation supports the first law. With the surfaces of equal temperature, called isotherms, drawn in a construction detail and the heat fluxes visualized by tracing the lines of equal flux, called isoflux lines, seen is that the last develop perpendicular to the isotherms, come closer and break up more where the isotherms do (Figure 1.1).

At the same time, in each material, the fluxes remain proportional to their thermal conductivity, which is often assumed to be a scalar and constant, even though for building and insulating materials its value depends on temperature, moisture content, sometimes the material's thickness and its age, while for anisotropic materials it becomes a tensor. Often, the term 'apparent' is therefore added to the supposedly constant value.

In right-angled Cartesian coordinates $[x, y, z]$, the heat flux along the three axes writes as:

$$q_x = \mathbf{q}_x \mathbf{u}_x = -\lambda \frac{\partial T}{\partial x} \quad q_y = \mathbf{q}_y \mathbf{u}_y = -\lambda \frac{\partial T}{\partial y} \quad q_z = \mathbf{q}_z \mathbf{u}_z = -\lambda \frac{\partial T}{\partial z}$$

In case °C instead of K is used, the heat flow across a surface dA with direction n equals:

$$\mathrm{d}\Phi_n = \mathbf{q}\mathbf{d}\mathbf{A}_n = -\lambda \frac{\partial \theta}{\partial n} \mathrm{d}A_n u_n^2 = -\lambda \frac{\partial \theta}{\partial n} \mathrm{d}A_n$$

Along each of the three axes, this gives as heat flow:

$$\mathrm{d}\Phi_x = -\lambda \frac{\partial \theta}{\partial X} \mathrm{d}A_x \quad \mathrm{d}\Phi_y = -\lambda \frac{\partial \theta}{\partial y} \mathrm{d}A_y \quad \mathrm{d}\Phi_z = -\lambda \frac{\partial \theta}{\partial Z} \mathrm{d}A_z$$

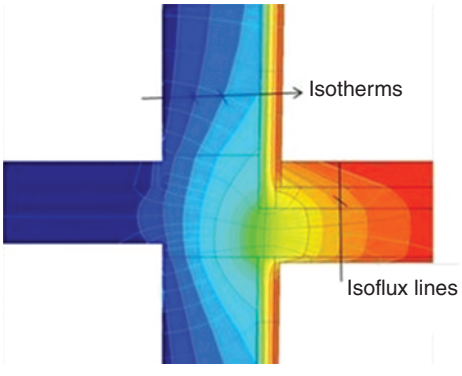


Figure 1.1 Floor with balcony traversing an outer wall insulated inside, lines of equal temperature and equal heat flux (the isotherms and isoflux lines).

1.2.2.2 Second Law

The second law is embedding the conduction equation in the conservation of energy one:

$$\mathrm{div}(\lambda\ \mathbf{grad}\ T) = \frac{\partial(\rho c T)}{\partial t} - \Phi' \tag{1.5}$$

This scalar relation allows calculating temperature fields. In case the thermal conductivity and the volumetric specific heat capacity are constant, the equation simplifies to what is called Fourier's second law:

$$\nabla^2 T = \left(\frac{\rho c}{\lambda}\right)\frac{\partial T}{\partial t} - \frac{\Phi'}{\lambda} \tag{1.6}$$

with ∇^2 the Laplace operator, in Cartesian coordinates equal to:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.7}$$

After a deeper digging in what the thermal conductivity represents at the material level, further discussion will focus on applying both laws to building assemblies.

1.2.3 Thermal Conductivity

1.2.3.1 In General

As mentioned, the property λ is not a constant. Its value in fact depends on the way heat is transferred in a material, which causes the already mentioned dependence on temperature, moisture content, sometimes thickness and age. This demands a closer look to how λ gets a value.

1.2.3.2 Heat Transfer Modes Fixing the Property

In dry porous materials four modes coexist (Figure 1.2): conduction along the material matrix, conduction through the pore gas, convection in the pore gas and radiation in each pore between its walls. When humid, two modes add: conduction in the water film against the pore walls and a latent heat exchange by evaporation at the warmer part of these walls, diffusion of the vapour so formed through the pore

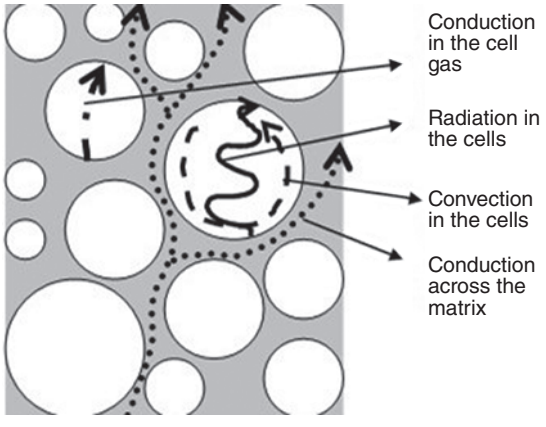


Figure 1.2 Heat flow modes in a porous material.

and condensation against the colder part of the pore walls. The value of a material's thermal conductivity depends on factors linked to these four or six modes.

If only conduction along the matrix and through the pore gas intervened, the thermal conductivity should equal:

$$\lambda_c = \lambda_M(1 - \Psi) + \lambda_G \frac{2\Psi}{1 + \Psi} \tag{1.8}$$

with λ_M the thermal conductivity of the matrix material, λ_G the thermal conductivity of the gas in the pores and Ψ the material's total porosity, equal to:

$$\Psi = (\rho_s - \rho)/\rho_s \tag{1.9}$$

with ρ the density and ρ_s the specific density of the matrix material, both in kg/m^3. Accordingly, the thermal conductivity should decrease the higher the total porosity. Or, a more porous material with lower density (ρ), must see its thermal conductivity drop, a fact experiments prove, see Figure 1.3. Material matrixes with lower thermal conductivity or gases in the pores insulating better than stagnant air should show a same trend. If pores could be vacuumed, the gas-related thermal conductivity should even turn zero ($\lambda_G \approx 0$).

In case other gases than air are filling the pores of a highly porous material, diffusion of air in, diffusion of part of these gases out or of being adsorbed by the material matrix will slowly increase the λ-value. In the first weeks after manufacturing, this increase obeys:

$$\lambda_c = \lambda_c(0) + C_1\sqrt{t}$$

where initially C_1 changes inversely proportional to the diffusion resistance factor μ of the matrix material and proportionally to the temperature with exponent $n < 1$ (T^n, T in K). Later the increase slows down to:

$$\lambda_c = \lambda_c(0) + (\lambda_c(\infty) - \lambda_c(0))[1 - \exp(C_2 t)]$$

with C_2 depending on the temperature and the diffusion resistance factor μ of the matrix material the same way as C_1 does, be it with values slackening the pace. Or, to

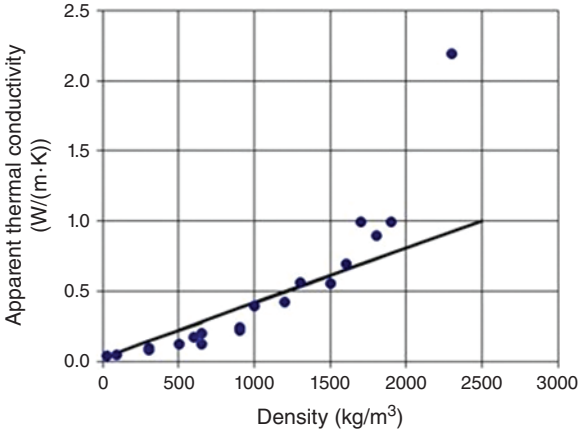


Figure 1.3 Thermal conductivity versus density. The full line stands for glass, from cellular to massive, $\lambda_M = 1$ W/(m·K), the dots for measured values for other materials.

more or less permanently store another gas than air in the pores, the matrix should be as vapour retarding as possible. An alternative for thermally insulating materials consists of covering the fresh boards with a vapour-tight lining. Or, to act as thermal insulation, materials need a very low density with, if the matrix is enough vapour retarding, a gas filling the pores that retards heat better than air does.

However, in larger pores, convection may develop. Its impact is quantified by multiplying the thermal conductivity of the pore gas with its convection-related Nusselt number ($X \geq 1$):

$$\lambda_c = \lambda_M(1-\Psi) + X\lambda_G\frac{2\Psi}{1+\Psi} \tag{1.10}$$

As a result, more heat will flow through the material than without convection. Or, looking to insulation materials, their pores should be small enough to keep the Nusselt number 1. Thus, an insulation material should not only be very porous, but it should also have small enough pores.

But additionally, in a cell where the walls that are not perfectly reflective and at different temperature radiant heat is exchanged giving a radiant term (λ_R) completing the λ_c-value just given:

$$\lambda = \lambda_m(1-\Psi) + X\lambda_G\frac{2\Psi}{1+\Psi} + F_{RC}\underbrace{\frac{4C_b T_m^3 d}{100^4\left(\frac{1}{e_1}+\frac{1}{e_2}+n\frac{1+\rho-\tau}{1-\rho+\tau}-1\right)}}_{\lambda_R} \tag{1.11}$$

with:

$$F_{RC} = 1 + 100^4\frac{\lambda_c(1/e_1+1/e_2-1)}{4C_b T_m^3 d}\left[\frac{\Delta\theta_1+\Delta\theta_n}{\Delta\theta/n}-1\right]$$

In case of insulation materials, in F_{RC}, the factor that corrects the thermal conductivity for the interaction between radiation, convection and conduction, is n standing for the number of pore walls, assumed parallel to the board's two faces, filling its thickness d. $\Delta\theta$ is the temperature difference between both faces, while $\Delta\theta_1$ and $\Delta\theta_n$ are the temperature differences between the linings that may cover both faces and the first pore wall encountered. In the λ_R formula, ρ and τ are the long wave reflectivity and transmissivity of these parallel pore walls, while for thermal insulation materials e_1 and e_2 are the long wave emissivity, side material of the linings covering both faces. As the radiant term λ_R is proportional to the third power of the temperature and the thermal conductivities of the matrix and the pore gas are temperature dependent, the overall relation with temperature may be rewritten as:

$$\lambda = \lambda_o + a_1\theta^n + a_2\theta^3 \tag{1.12}$$

with $0 < n < 1$. Between -20 and $50\,°C$, that equation becomes $\pm$ linear:

$$\lambda = \lambda_o + a_R\theta = \lambda_o\left(1 + a'_R\theta\right) \tag{1.13}$$

The lighter a highly porous material, meaning the thinner the pore walls or the larger the pores, the higher a_R and the more temperature dependant its thermal conductivity. Because few large than small pores fit into a given layer thickness and

because thin pore walls show a higher long wave transmissivity than thicker ones, radiation gains importance in equally heavy materials having as well small or large pores. But the thermal conductivity must increase with a board's thickness. In fact, with the ratio between thickness (d) and mean pore width (d_P) replacing the number of pore walls, thickness appears in the numerator of λ_R but stays hidden in the denominator. This way, the radiation term (λ_R) shifts from 0 for a zero thickness to an asymptotic value ($\lambda_{R\infty}$) when infinitely thick:

$$\lambda_{R\infty} \approx \frac{4FC_b T_m^3 d_P}{100^4 \left[\frac{1+\rho-\tau}{1-\rho+\tau}\right]} \tag{1.14}$$

That asymptote increases for larger pores and pore walls transmitting more radiation, thus for materials with lower density. Or, radiant exchange prevents the thermal conductivity to continually drop with lower density. In fact, once below a certain value, further decrease demands larger pores or thinner pore walls. In both cases, radiation increases, turning the thermal conductivity into a sum of a monotonously decreasing conductive and an increasing radiant part. Or, surely for thermally insulating materials with given thickness, the λ-value must at some density pass a minimum (Figure 1.4):

$$\lambda = b_1 + b_2\rho + b_3/\rho \tag{1.15}$$

In a moist material, heat conduction in the adsorbed water layers and condensed water islands adds. For open porous materials, the result is a linear relation between thermal conductivity and moisture ratio, be it that for very porous insulating materials, the relation, now with the volumetric moisture ratio, is rather parabolic:

$$\lambda = \lambda_d(1 + a_X X) \quad \lambda = \lambda_d(1 + a_\Psi \Psi + b_\Psi \Psi^2) \tag{1.16}$$

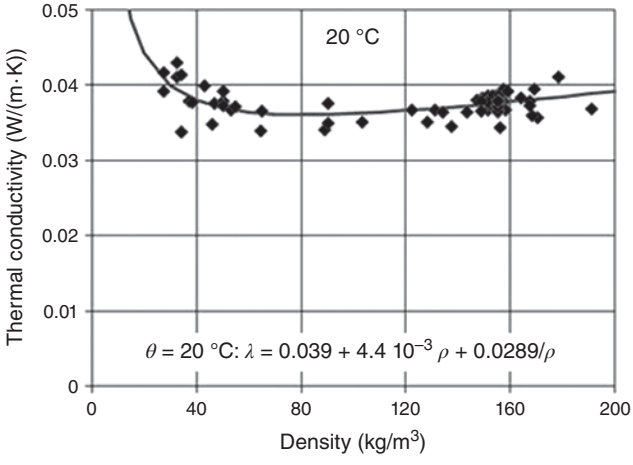


Figure 1.4 Mineral wool, thermal conductivity versus density.

In both, λ_d is the thermal conductivity when dry. Of course, porous materials will also see latent heat exchanged, adding as heat flux:

$$q_v = l_b g_v = -\frac{l_b}{\mu N}\frac{dp_{sat}}{d\theta}\text{grad}\theta \tag{1.17}$$

with l_b the heat of evaporation. As a result, the thermal conductivity of a moist material becomes:

$$\lambda = \lambda_d(1 + a_X X)\left(1 + \frac{l_b}{\mu N \lambda_d(1 + a_X X)}\frac{dp_{sat}}{d\theta}\right) \tag{1.18}$$

Due to the latent heat exchange, temperature affects the thermal conductivity of wet materials more than radiation does. The impact quickly increases with a lower vapour resistance factor of the matrix. Fibrous insulation materials such as mineral wool suffer the most. Evaporation at the warm side is giving a sudden boost to their thermal conductivity (Figure 1.5).

For simplicity reasons, in what follows, except if otherwise said, the 'apparent' thermal conductivity is considered constant.

1.2.4 Steady-State

1.2.4.1 What?

In steady state, the temperatures, the heat fluxes, the material properties and energy dissipation, if any, remain time independent. If so, the derivative of the temperature over time turns zero:

$$\partial T/\partial t = 0$$

With the temperature in °C, Fourier's second law then simplifies to:

$$\nabla^2\theta = -\Phi'/\lambda \tag{1.19}$$

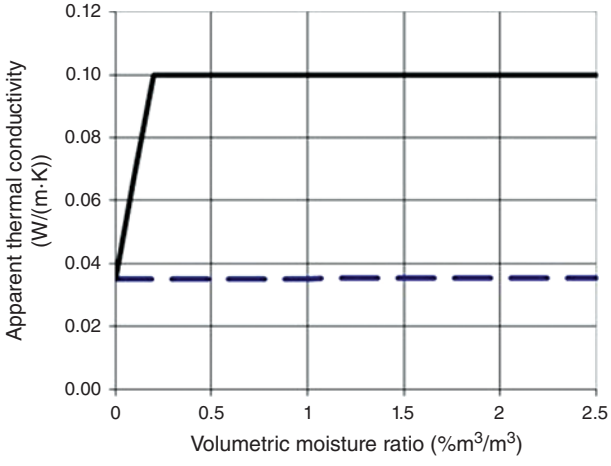


Figure 1.5 Mineral wool, thermal conductivity versus volumetric moisture ratio, a full line for the warm, a dotted for the cold side initially moist.

1.2.4.2 One Dimension, Flat Assemblies

1.2.4.2.1 In General

With the temperature change perpendicular to both end faces of a flat assembly, the second law further reduces to:

$$\frac{d^2\theta}{dx^2} = -\frac{\Phi'}{\lambda} \tag{1.20}$$

Without dissipation, this becomes:

$$d^2\theta/dx^2 = 0$$

with as solution: $\theta = C_1 x + C_2$ where C_1 and C_2 are the integration constants, fixed by the boundary conditions. This really simple equation governs heat conduction across flat assemblies with both sides at constant but different temperature. Most buildings are composed of flat assemblies, take low-slope roofs, sloped roof pitches, outer walls, floors, party walls, partitions, glass surfaces, etc. Their cross-section is either single- or multi-layered.

1.2.4.2.2 Single-Layered Walls

Supposedly known are the thermal conductivity and the thickness (d) of the material used. The boundary conditions are $x = 0$: $\theta = \theta_{s1}$; $x = d$: $\theta = \theta_{s2}$ with θ_{s1} and θ_{s2} the surface temperatures of both end faces. If end face s_1 is colder than end face s_2, the integration constants become:

$$C_2 = \theta_{s1} \quad C_1 = (\theta_{s2} - \theta_{s1})/d$$

This gives as temperature course through the assembly:

$$\theta = \frac{\theta_{s2} - \theta_{s1}}{d} x + \theta_{s1} \tag{1.21}$$

Or, in steady state and without heat source or sink, the temperatures in a single-layer assembly vary linearly from the warmer to the colder end face (Figure 1.6).

The heat flux becomes:

$$\mathbf{q} = -\lambda\, \mathbf{grad}\, \theta = -\lambda \frac{d\theta}{dx} = -\lambda \frac{\theta_{s1} - \theta_{s2}}{d} \tag{1.22}$$

Figure 1.6 Single-layer assembly, temperatures.

Or, in absolute terms:

$$q = \lambda(\theta_{s2} - \theta_{s1})/d \tag{1.23}$$

The constant flux so develops proportionally to the thermal conductivity of the material and the difference in surface temperature of both end faces but inversely proportional to the assembly's thickness. With the difference and thickness given, a lower thermal conductivity reduces the heat flux, meaning less heat lost or gained. Materials with very low thermal conductivity are therefore named '(thermally) insulating'. Reshuffling the heat flux equation gives:

$$q = \frac{\Delta\theta}{d/\lambda} \tag{1.24}$$

with d/λ the thermal resistance of such flat, single-layer assembly, symbol R, units $m^2 \cdot K/W$.

The higher R, the lower the heat flux for a given difference in surface temperatures or, the better insulating the assembly. That higher requires either a thicker assembly or the use of a better insulating material with same thickness. The inverse of the thermal resistance is called the thermal conductance, symbol P, units $W/(m^2 \cdot K)$, a quantity telling how much heat per unit of time and surface passes through an assembly at 1 °C difference between both end faces.

1.2.4.2.3 Multi-Layered Assemblies

Multi-layer assemblies, as most floors, walls and roofs in buildings are, consist of two or more plan-parallel material layers. An example is the cavity wall of Figure 1.7 with concrete block inside leaf, cavity fill and concrete block veneer.

In steady state and without local heat sources and sinks, the heat flux through must be the same in all layers. With the temperature θ_{s1} on end face 1 higher than θ_{s2} on end face 2, the thermal conductivities and thicknesses of all layers known and the contact resistances in between negligible, the constant heat flux **q** must equal:

$$\text{Layer 1} \quad q = \lambda_1 \frac{\theta_1 - \theta_{s1}}{d_1}$$

Figure 1.7 Filled cavity wall.

$$\text{Layer } 2 \quad q = \lambda_2 \frac{\theta_2 - \theta_1}{d_2}$$

$$\text{Layer } n-1 \quad q = \lambda_{n-1} \frac{\theta_{n-1} - \theta_{n-2}}{d_{n-1}}$$

$$\text{Layer } n \quad q = \lambda_n \frac{\theta_{s2} - \theta_{n-1}}{d_n}$$

with θ_1, θ_2, …, θ_{n-1} the unknown interface temperatures. Rearrangement and summing give:

$$\begin{aligned} q\frac{d_1}{\lambda_1} &= \theta_1 - \theta_{s1} \\ + q\frac{d_2}{\lambda_2} &= \theta_2 - \theta_1 \\ + \cdots & \\ + q\frac{d_n}{\lambda_n} &= \theta_{s2} - \theta_{n-1} \\ \hline q\sum_{i=1}^{n}\left(\frac{d_i}{\lambda_i}\right) &= \theta_{s2} - \theta_{s1} \end{aligned} \tag{1.25}$$

or:

$$q = \frac{\theta_{s2} - \theta_{s1}}{\sum_{i=1}^{n}(d_i/\lambda_i)}$$

$\sum(d_i/\lambda_i)$, symbol R_T, units m^2K/W, is called the total thermal resistance of the multi-layer assembly, while the ratios d_i/λ_i, symbol R_i, same units, represent the thermal resistances of the separate layers. The higher the total thermal resistance, the lower the steady state heat flux and the better insulating the assembly. A high total thermal resistance R_T requires the inclusion of a sufficiently thick insulation layer. After the energy crises of 1973 and 1979, lowering the net heating demand by insulating the multi-layer assemblies, building envelopes are composed of, finally became a legal requirement. Since the 2000s, limiting the CO_2 release when heating with fossil fuels, more, moving to carbon neutrality added extra pressure to excellently insulate the envelope assemblies in new construction and, if doable, in renovation.

The layers that make up an assembly fix its total thermal resistance. Being the sum of the thermal resistances of all, the commutation rule applies, meaning layer sequence has no impact on that value. So, inside insulation should not differ from outside insulation. Of course, from an overall building physics point of view, this is untrue: a same total thermal resistance yes, but a diverging overall heat, air and moisture performance.

With the surface temperatures on both end faces known, the temperatures in all consecutive interfaces follow from reshuffling layer per layer the heat flux equation:

$$\theta_1 = \theta_{s1} + q\frac{d_1}{\lambda_1} = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{R_1}{R_T}$$

$$\theta_2 = \theta_1 + q\frac{d_2}{\lambda_2} = \theta_1 + qR_2 = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{(R_1 + R_2)}{R_T}$$

$$\theta_{n-1} = \theta_{n-2} + q\frac{d_{n-1}}{\lambda_{n-1}} = \theta_{s1} + (\theta_{s2} - \theta_{s1})\frac{\sum_{i=1}^{n-1} R_i}{R_T}$$

Writing $\theta_i = \theta_{i-1} + qd_i/\lambda_i$ as $(\theta_i - \theta_{i-1})/d_i = q/\lambda_i$ underlines that the temperature gradient in a layer is inversely proportional to its thermal conductivity. Hence, gradients are large in insulating layers and small in conductive layers. Each layer equation can be rewritten now as:

$$\theta_x = \theta_{s1} + qR_{s1}^x \tag{1.26}$$

with R_{s1}^x the thermal resistance between end face s1 and the considered interface x. If calculating should start at end face s2, that equation becomes:

$$\theta_x = \theta_{s2} - qR_{s2}^x \tag{1.27}$$

In an axis system with the thermal resistance R as apsis and the temperature θ as ordinate, both equations give a straight line between the temperatures on both end faces ((0, θ_{s1}) and (R, θ_{s2})) having the heat flux as slope. Or, a multi-layer assembly looks in this axis system as if single-layered. To construct the temperature line, the assembly is therefore redrawn with each layer as thick as its thermal resistance. Then, the surface temperature θ_{s1} on end face s1 and the surface temperature θ_{s2} on end face s2 are marked and the line linking both traced. The temperature versus thickness curve then follows from transposing the intersections between that temperature line and the successive interfaces to the same interface on the thickness graph and linking the consecutive temperatures with line segments, see Figure 1.8.

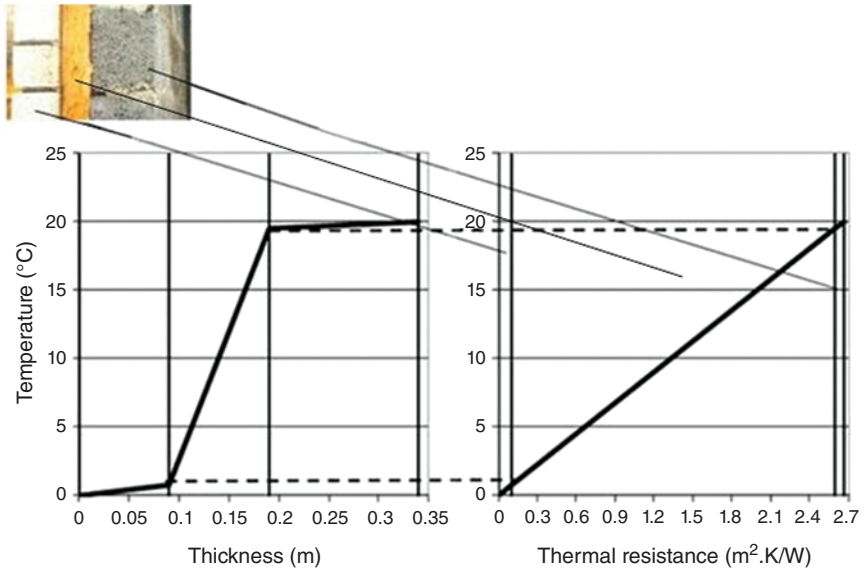


Figure 1.8 Temperatures in a composite assembly, graphical construction.

For single- and multi-layer assemblies, the heat flow through follows from the product of the heat flux with their surface (A) in m^2:

$$\Phi = qA \tag{1.28}$$

1.2.4.2.4 Electrical Analogy

The current passing through a series connection of electrical resistors (R_{ei}) subjected to a voltage differential ΔV is

$$i = \Delta V / \sum R_{ei} \tag{1.29}$$

Or; voltage so to say replaces the temperature, the current the heat flux and the resistors the thermal resistances. This allows converting heat conduction problems into an electrical analogy.

1.2.4.2.5 Special Cases

The first concerns a single-layer assembly, of which the thermal conductivity along its thickness (x) changes with the local temperature or with moisture content. Should its value depend linearly on temperature ($\lambda = \lambda_o + a\theta$), then the heat flux through becomes ($x = 0$, $\theta = \theta_{s1}$; $x = d$, $\theta = \theta_{s2}$):

$$\int_{\theta_{s1}}^{\theta_{s2}} (\lambda_0 + a\theta)\, \mathrm{d}\theta = \int_0^d q\, \mathrm{d}x$$

This gives as solution:

$$\lambda_0(\theta_{s2} - \theta_{s1}) + \frac{a\left(\theta_{s2}^2 - \theta_{s1}^2\right)}{2} = qd \quad \text{or} \quad q = \lambda(\theta_m)\frac{\theta_{s2} - \theta_{s1}}{d} \tag{1.30}$$

with $\lambda(\theta_m)$ the thermal conductivity at the single-layer's average temperature. The thermal resistance so is $d/\lambda(\theta_m)$, while the temperature course turns into a parabolic relation:

$$a\theta^2/2 + \lambda_0\theta = q\,x + C$$

For an assembly with locally varying moisture content $w(x)$ (Figure 1.9) making the thermal conductivity a function of $w(x)$, thus of x, the heat flux through is ($x = 0$: $\theta = \theta_{s1}$; $x = d$: $\theta = \theta_{s2}$):

$$q\int_0^d \frac{\mathrm{d}x}{\lambda(x)} = \int_{\theta_{s1}}^{\theta_{s2}} \mathrm{d}\theta$$

Is the moisture content distributed in a way the thermal conductivity increases linearly with x ($\lambda = \lambda_o + ax$), solving the integrals then gives:

$$q = \frac{\theta_{s2} - \theta_{s1}}{\frac{1}{a}\ \ln\left(\frac{\lambda_0 + a\,d}{\lambda_0}\right)} \tag{1.31}$$

Again, the denominator stands for the thermal resistance. The temperature curve becomes:

$$\theta_x = \theta_{s1} + q\left[\frac{1}{a}\ln\left(\frac{\lambda_0 + a\,x}{\lambda_0}\right)\right] \tag{1.32}$$

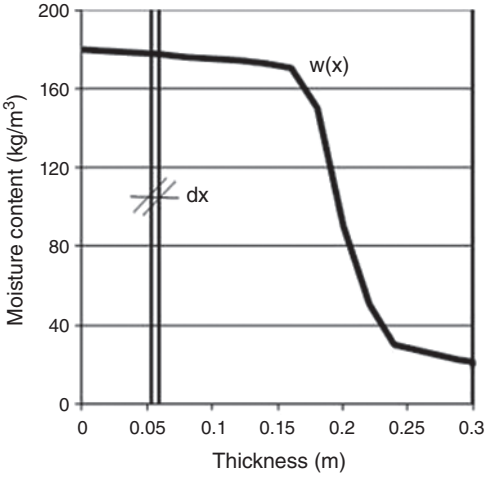


Figure 1.9 Moisture profile turning thermal conductivity into a function of the ordinate *x*, representing the thickness.

The second applies to a single-layer assembly dissipating or absorbing Φ' joule of heat per unit of time and volume. If spread uniformly over its thickness, the steady state balance becomes:

$$\frac{d^2\theta}{dx^2} = -\frac{\Phi'}{\lambda}$$

With as boundary conditions $x = 0$: $\theta = \theta_{s1}$; $x = d$: $\theta = \theta_{s2}$ ($\theta_{s1} < \theta_{s2}$), the solution so is:

$$\theta = -\frac{\Phi'}{2\lambda}x^2 + C_1 x + C_2 \tag{1.33}$$

This parabolic temperature curve turns convex when heat is dissipated and concave when heat is absorbed by the assembly (Figure 1.10).

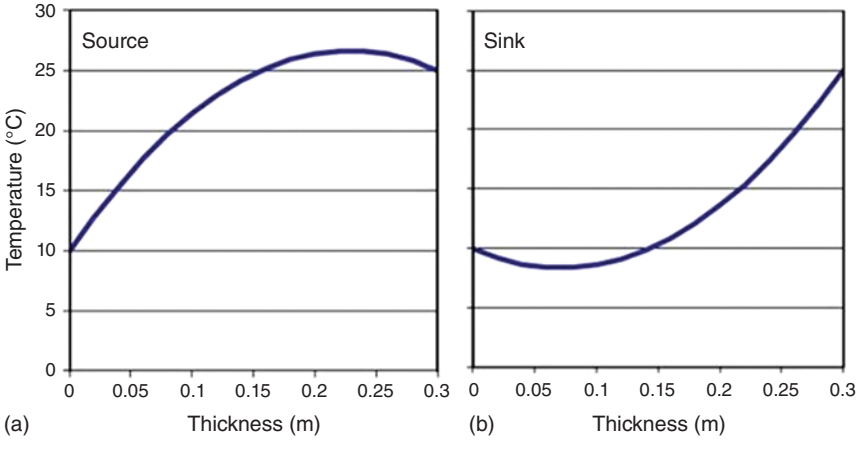


Figure 1.10 A single-layered assembly acting as uniformly distributed heat source (a) or sink (b), temperature curves.

Related heat flux becomes:

$$q = -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x} = \Phi' x - C_1 \lambda \tag{1.34}$$

Opposite to no heat dissipated, the flux yet changes along the assembly's thickness. The boundary conditions give as integration constant:

$$C_1 = \frac{\theta_{s2} - \theta_{s1}}{d} + \frac{\Phi d'}{2\lambda} \quad C_2 = \theta_{s1}$$

The third concerns a multi-layer assembly with locally a heat source or sink, for example due to condensation in or drying from an interface x. The heat dissipated or absorbed per s is q', while the temperature θ_{s1} on one end face is higher than the temperature θ_{s2} on the other. A steady state heat balance for interface x then allows calculating the overall temperature curve. Assuming that heat flows from both end faces to interface x, related fluxes become:

$$\text{From end face } s1 \text{ to } x: \quad q^x_{s1} = \frac{\theta_{s1} - \theta_x}{R^x_{s1}}$$

$$\text{From end face } s2 \text{ to } x: \quad q^x_{s2} = \frac{\theta_{s2} - \theta_x}{R^x_{s2}}$$

In both equations, θ_x is the unknown temperature in x. Setting the difference between the two equals to the heat released (+) or absorbed (−) per unit of time gives for θ_x:

$$\theta_x = \frac{R^x_{s2}\theta_{s1} + R^x_{s1}\theta_{s2} + q' R^x_{s1} R^x_{s2}}{R^x_{s1} + R^x_{s2}} \tag{1.35}$$

Including this result in both flux equations gives:

$$q^x_{s1} = \left(\frac{\theta_{s1} - \theta_{s2}}{R_T}\right) - \frac{q' R^x_{s2}}{R_T} \quad q^x_{s2} = -\left[\left(\frac{\theta_{s1} - \theta_{s2}}{R_T}\right) + \frac{q' R^x_{s1}}{R_T}\right] \tag{1.36}$$

with R_T the total thermal resistance of the assembly: $R_T = R^x_{s1} + R^x_{s2}$. A heat source so gives a lower in- and higher outgoing flux, a sink the inverse. With the thermal resistance as apsis, the temperature curve consists of two straight lines with different slope, linking the temperatures on both end faces to the one in interface x (Figure 1.11).

1.2.4.3 Two Dimensions, Cylinder Symmetric

In cylindric coordinates, pipes behave as if one-dimensional. Considered is a hanged heating pipe with inside radius r_1 and outside radius r_2. Of interest is the heat loss per running metre the pipe's temperature and the insulation efficiency. The inner face has as temperature θ_{s1}, and the outer as temperature θ_{s2} (Figure 1.12).

In steady state and without dissipation other than from the liquid or gas transported, the heat flow passing each concentric cylinder in the pipe must be identical. With the centre as origin, the flow per running metre so must equal:

$$\Phi = -\lambda\,(2\pi r)\mathrm{d}\theta/\mathrm{d}r = C^t$$

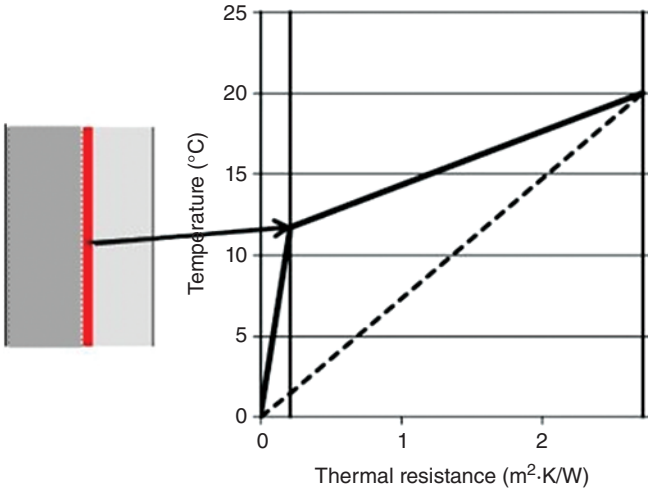


Figure 1.11 Two-layer assembly, heat source in the interface between the two.

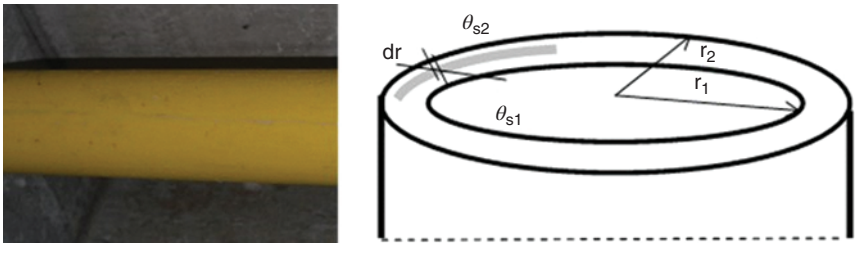


Figure 1.12 The pipe problem.

Integration gives:

$$\Phi \int_{r_1}^{r_2} \frac{\mathrm{d}r}{r} = -2\pi\lambda \int_{\theta_{s1}}^{\theta_{s2}} \mathrm{d}\theta \quad \text{or} \quad \Phi = \frac{\theta_{s1} - \theta_{s2}}{\left(\frac{\ln(r_2/r_1)}{2\pi\lambda}\right)} \tag{1.37}$$

The denominator, called the thermal resistance of the pipe per running metre units m·K/W, can be seen as the equivalent of the thermal resistance of a flat assembly. For a composite pipe, a same reasoning as for a multi-layer flat assembly gives as heat flow per running metre:

$$\Phi = \frac{\theta_{s1} - \theta_{s2}}{\sum_{i=1}^{n} \left[\frac{\ln(r_{i+1}/r_i)}{2\pi\lambda_i}\right]} \tag{1.38}$$

The temperatures follow from:

$$\theta_{i+1} = \theta_{s1} + \Phi \sum_{i=1}^{i} \left[\frac{\ln(r_{i+1}/r_i)}{2\pi\lambda_i}\right] \tag{1.39}$$

1.2.4.4 Two and Three Dimensions: Thermal Bridges

When looking in detail to outer walls, roofs, floors and partitions, the assumption 'flat' does not apply everywhere. What about lintels above windows? What about window reveals? What about junctions between two outer walls? What about corners between two outer walls and a low-slope roof (Figure 1.13)? All act as thermal bridges. Even both end faces of a flat assembly could be non-isothermal, turning its thermal picture from one to two or three dimensionally.

Studying the steady state heat transfer through, now requires a return to the second law:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=\pm\frac{\Phi'}{\lambda}$$

or, without heat dissipation:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=0$$

For some very simple combinations - one material, simple geometry, simple boundary conditions - this partial differential equation can be solved analytically. However, in the majority of building-related cases, the use of control volumes (CVM) is preferred. Many building details consist of rectangular material volumes, which are easily meshed in cube or beam-like control volumes. In steady state and if three-dimensional, the heat flow from the six adjacent volumes to the central must be 0 then (Figure 1.14). If meshing gives control volumes whose sides coincide with the interfaces between layers, all are material-homogenous, but then the calculations will not give the interface temperatures. Preference therefore goes to control volumes with the centre on interfaces between materials. Related lack of homogeneity is compensated by the fact the calculations give the interface temperatures then.

In case all control volumes are cubic, then the distances to consider along the x-, y- and z-axis around each equal the one between its centre and that of the six adjacent control volumes. If additionally, all lay in the same material, then, with the named distance called the mesh width a, the heat flow from the adjacent control volume $(l{-}1, m, n)$ to the central (l, m, n) equals:

$$\Phi_{l-1,m,n}^{l,m,n}=\lambda\frac{(\theta_{l-1,m,n}-\theta_{l,m,n})\,a^2}{a}=a\lambda(\theta_{l-1,m,n}-\theta_{l,m,n})$$

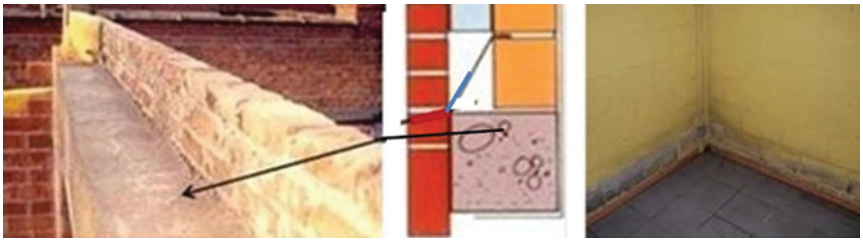


Figure 1.13 Lintel above a window and corner between two outside walls and the floor.

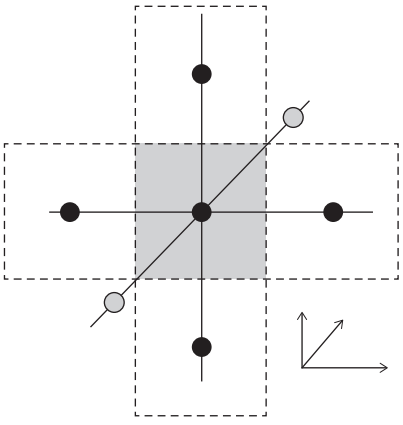


Figure 1.14 Central and adjacent control volumes.

For the other five, the flows become:

$$\Phi^{l,m,n}_{l+1,m,n} = \lambda \frac{(\theta_{l+1,m,n} - \theta_{l,m,n})\, a^2}{a} = a\lambda(\theta_{l+1,m,n} - \theta_{l,m,n})$$

$$\Phi^{l,m,n}_{l,m-1,n} = \lambda \frac{(\theta_{l,m-1,n} - \theta_{l,m,n})\, a^2}{a} = a\lambda(\theta_{l,m-1,n} - \theta_{l,m,n})$$

$$\Phi^{l,m,n}_{l,m+1,n} = \lambda \frac{(\theta_{l,m+1,n} - \theta_{l,m,n})\, a^2}{a} = a\lambda(\theta_{l,m+1,n} - \theta_{l,m,n})$$

$$\Phi^{l,m,n}_{l,m,n-1} = \lambda \frac{(\theta_{l,m,n-1} - \theta_{l,m,n})\, a^2}{a} = a\lambda(\theta_{l,m,n-1} - \theta_{l,m,n})$$

$$\Phi^{l,m,n}_{l,m,n+1} = \lambda \frac{(\theta_{l,m,n+1} - \theta_{l,m,n})\, a^2}{a} = a\lambda(\theta_{l,m,n+1} - \theta_{l,m,n})$$

Adding the six and sum zero gives:

$$\theta_{l-1,m,n} + \theta_{l+1,m,n} + \theta_{l,m-1,n} + \theta_{l,m+1,n} + \theta_{l,m,n-1} + \theta_{l,m,n+1} - 6\,\theta_{l,m,n} = 0$$

Unknown in this linear equation are the temperatures in the central and six adjacent control volumes. In two dimensions, each control volume has four adjacent, giving as sum:

$$\theta_{l-1,m} + \theta_{l+1,m} + \theta_{l,m-1} + \theta_{l,m+1} - 4\,\theta_{l,m} = 0$$

When the central control volume bridges an interface between two materials with thermal conductivity λ_1 and λ_2 parallel to the $[x, y]$ plane, then, for a mesh width 'a' along the three axes, the heat flow from an adjacent control volume $(l{-}1, m, n)$ to the central (l, m, n) changes to:

$$\Phi^{l,m,n}_{l-1,m,n} = \lambda_1 \frac{(\theta_{l-1,m,n} - \theta_{l,m,n})\, a^2}{2a} + \lambda_2 \frac{(\theta_{l-1,m,n} - \theta_{l,m,n})\, a^2}{2a}$$

or:

$$\Phi^{l,m,n}_{l-1,m,n} = \frac{a(\lambda_1 + \lambda_2)(\theta_{l-1,m,n} - \theta_{l,m,n})}{2}$$

For the other five, that flow becomes:

$$\Phi^{l,m,n}_{l+1,m,n} = \frac{a(\lambda_1 + \lambda_2)(\theta_{l+1,m,n} - \theta_{l,m,n})}{2}$$

$$\Phi^{l,m,n}_{l,m-1,n} = \frac{a(\lambda_1 + \lambda_2)(\theta_{l,m-1,n} - \theta_{l,m,n})}{2}$$

$$\Phi^{l,m,n}_{l,m+1,n} = \frac{a(\lambda_1 + \lambda_2)(\theta_{l,m+1,n} - \theta_{l,m,n})}{2}$$

$$\Phi^{l,m,n}_{l,m,n-1} = a\lambda_1(\theta_{l,m,n-1} - \theta_{l,m,n})$$

$$\Phi^{l,m,n}_{l,m,n+1} = a\lambda_2(\theta_{l,m,n+1} - \theta_{l,m,n})$$

Adding and sum zero gives:

$$\frac{(\lambda_1 + \lambda_2)(\theta_{l-1,m,n} + \theta_{l+1,m,n} + \theta_{l,m-1,n} + \theta_{l,m+1,n})}{2} + \lambda_2\theta_{l,m,n-1} + \lambda_1\theta_{l,m,n+1} - 3(\lambda_1 + \lambda_2)\,\theta_{l,m,n} = 0$$

Again, the result is a linear equation with: the temperatures in the central and the six adjacent control volumes unknown. In two dimensions, the result is:

$$\frac{(\lambda_1 + \lambda_2)(\theta_{l-1,m} - \theta_{l+1,m})}{2} + \lambda_2\theta_{l,m-1} + \lambda_1\theta_{l,m+1} - 2(\lambda_1 + \lambda_2)\,\theta_{l,m,n} = 0$$

In case the central control volume lays on the intersection between three materials with thermal conductivity λ_1, λ_2 and λ_3 and interfaces parallel to the $[x, y]$ and $[y, z]$-plane, the sum changes to:

$$(\lambda_2 + \lambda_3)\frac{\theta_{l-1,m,n}}{2} + (\lambda_2 + \lambda_3)\frac{\theta_{l+1,m,n}}{2} + \lambda_3\theta_{l,m-1,n} + (\lambda_1 + \lambda_2)\frac{\theta_{l,m+1,n}}{2} + (\lambda_1 + \lambda_2 + \lambda_3)\frac{\theta_{l,m,n-1} - \theta_{l,m,n+1}}{4} - (3\lambda_1 + 3\lambda_2 + 6\lambda_3)\frac{\theta_{l,m,n}}{2} = 0$$

Again, a linear equation with: seven unknowns. Two dimensions give:

$$\lambda_3\theta_{l-1,m} + (\lambda_1 + \lambda_2)\frac{\theta_{l+1,m}}{2} + (\lambda_2 + \lambda_3)\frac{\theta_{l,m-1}}{2} + (\lambda_1 + \lambda_3)\frac{\theta_{l,m+1}}{2} - (\lambda_1 + \lambda_2 + \lambda_3)\,\theta_{l,m} = 0$$

All other cases are solved the same way. In three dimensions, a control volume could contain up to eight materials, in two up to four. All generate a linear equation with seven or five unknowns. For p control volumes, the result is a system of p equations with p unknown temperatures, in which the known temperatures figure as boundary condition. Solving gives the temperature distribution in the building detail considered. Once known, the above equations allow calculating the heat fluxes along the three axes between the control volume centres.

Generalizing the algorithm begins with calling the surface-linked thermal conductance between adjacent control volumes P_s, units W/K. If both consist of the same material, then P_s becomes:

$$P_s = (\lambda/d)\,A$$

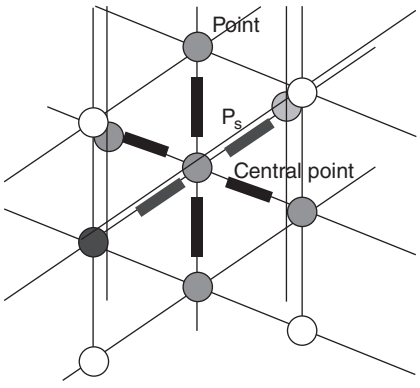


Figure 1.15 Summing up the six thermal conductance's between the central point considered and the six neighbour ones.

If the two belong to different materials, the resulting surface-linked conductance P_s turns into a serial, a parallel or a serial/parallel circuit of conductance, to be calculated in advance.

$$\Phi^{l,m,n}_{1-l,m,n} = P^{l,m,n}_{s1-l,m,n}(\theta_{1-l,m,n} - \theta_{l,m,n}) \tag{1.40}$$

Summing up the related heat flows gives per central point (Figure 1.15):

$$\sum_{\substack{i=l,m,n \\ j=\pm 1}} [P_{s,i+j}\theta_{i+j}] - \theta_{l,m,n} \sum_{\substack{i=l,m,n \\ j=\pm 1}} P_{s,i+j} = 0 \tag{1.41}$$

After transferring the known temperatures to the right, the system representing the two or three dimensional detail transposed into p control volumes converts to:

$$[P_s]_{p,p}[\theta]_p = [P_{s,i,j,k}\theta_{i,j,k}]_p$$

with $[P_s]_{p,p}$ the p rows, p columns matrix of the conductance's, $[\theta]_p$ the column matrix of the p unknown temperatures and $[P_{s,i,j,k}\theta_{i,j,k}]_p$ the column matrix of the known temperatures. Solving gives the temperature distribution in the detail, which then allows to calculate the heat flow passing through.

The accuracy of a CVM calculation depends on how fine is meshed. The finer, the more the solution will approximate reality. An infinitely fine mesh should produce the exact solution, but then, a system with an infinite number of equations has to be solved, which would last infinitely. Therefore, a compromise between accuracy and processing time includes making the mesh less fine where small and finer where large temperature gradients are expected. Today, powerful software packages to calculate the 2D or 3D heat transfer through building details are available. In former times, before computer software took over, electrical analogies were used.

1.2.5 Non-steady-state

1.2.5.1 In General

Opposite to what was assumed till now, due to varying boundary conditions, varying material properties and/or the presence of varying internal heat sources or sinks the temperatures in and heat fluxes through assemblies are mostly time dependent.

If nonetheless the material properties can be considered constant and no heat sources or sinks intervene, the second law becomes:

$$\nabla^2\theta = \frac{\rho c}{\lambda}\frac{\partial\theta}{\partial t}$$

The temperatures and heat fluxes on both end faces of an assembly now may vary periodically or suddenly, periodically when linked to the air temperature outdoors, which fluctuates on a yearly, *n* days, daily, hourly or even a shorter time basis, suddenly when warming up a space (Figure 1.16).

1.2.5.2 Periodic Boundary Conditions, Flat Assemblies

1.2.5.2.1 Methodology

Setting $\lambda/(\rho c) = a$ reduces the second law to:

$$a\frac{\partial^2\theta}{\partial x^2} = \frac{\partial\theta}{\partial t}$$

The ratio a, units m^2/s, is called the thermal diffusivity of a material, a characteristic indicating how easily a local temperature change spreads in it. The higher a, the faster this happens.

A high thermal diffusivity requires a light material with high thermal conductivity or a heavy material with low volumetric heat capacity. None of the two exists. Light materials have a low thermal conductivity, and heavy materials have a high volumetric heat capacity. Very different materials can so have quite similar thermal diffusivities. An exception is the metals.

Substituting the thickness x in the equation by the layer's thermal resistance $R\ (=x/\lambda)$, which means multiplying the left and right with λ^2, gives as temperature and heat flux:

$$\frac{\partial^2\theta}{\partial R^2} = \rho c\lambda\frac{\partial\theta}{\partial t} \qquad q = -\partial\theta/\partial R \tag{1.42}$$

Assumed now is that the temperatures on both end faces of a flat layer fluctuate with period T. Transposing these fluctuations into a Fourier series gives:

$$\theta_s = \frac{B_{s0}}{2} + \sum_{n=1}^{\infty}\left[A_{sn}\sin\left(\frac{2n\pi t}{T}\right) + B_{sn}\cos\left(\frac{2n\pi t}{T}\right)\right]$$

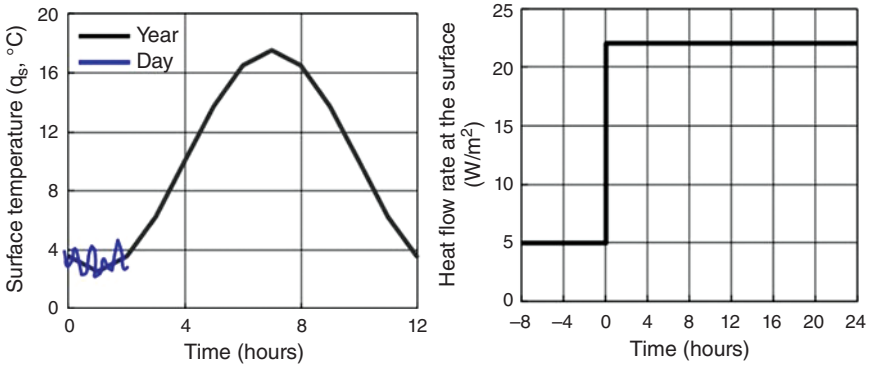


Figure 1.16 A periodic change and a sudden jump called a step change.

with:

$$A_{sn} = \frac{2}{T}\int_0^T \theta_s(t)\ \sin\left(\frac{2n\pi t}{T}\right)\ \mathrm{d}t \qquad B_{sn} = \frac{2}{T}\int_0^T \theta_s(t)\ \cos\left(\frac{2n\pi t}{T}\right)\ \mathrm{d}t$$

The value $B_{so}/2$ represents the average temperature on the end face considered over the period T, while the values A_{s1}, A_{s2}, …, A_{sn}, B_{s1}, B_{s2}, …, B_{sn} stay for the harmonics of the 1st, 2nd, …, nth order. Rewriting the series in complex form using Euler's formulas for sin(x) and cos(x) gives:

$$\text{Euler}: \quad \sin(x) = \frac{\exp(ix) - \exp(-ix)}{2i}$$
$$\cos(x) = \frac{\exp(ix) + \exp(-ix)}{2} \qquad x = 2n\pi t/T$$

$$\text{Result}: \theta_s(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha_{sn}\ \exp\left(\frac{2in\pi t}{T}\right)\right]$$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[(A_{sn} + iB_{sn})\ \exp\left(\frac{2in\pi t}{T}\right)\right] \tag{1.43}$$

α_{sn} being the nth complex temperature having as amplitude and phase shift:

Amplitude	Phase shift
$\sqrt{B_{sn}^2 + A_{sn}^2}$	$\mathrm{a\,tan}(-\mathrm{A_{sn}}/\mathrm{B_{sn}})$

The amplitude tells how large half the temperature variation is, and the phase shift how long the time delay is compared to a cosine with period $T/(2n\pi)$ (in radians). For the monthly mean air temperature outdoors at Uccle of Figure 1.17, the thick proxy temperature line has as equation:

$$\theta_e = 9.45 + 7.18\ \cos\left(\frac{2\pi t}{365.25} - (-2.828)\right)$$

with t the time in days.

Included the leap years, a year counts 365.25 days. The annual average temperature (θ_{em}) so is 9.45 °C the annual amplitude (θ_{e1}) 7.18 °C, and the phase shift in radians −2.828. The time-dependent term resembles a rotating vector with value 7.18 starting −2.828 radians away from the real axis. Conversion into a Fourier series gives:

$$\theta_e = \frac{B_o}{2} + \theta_{e1}\left[A_1\ \sin\left(\frac{2\pi t}{365.25}\right) + B_1\ \cos\left(\frac{2\pi t}{365.25}\right)\right]$$

or, with $B_o = 18.9$, $A_1 = \theta_{e1}\sin(\varphi_{e1}) = -2.21$ and $B_1 = \theta_{e1}\cos(\varphi_{e1}) = -6.83$:

$$\theta_e = 9.45 + -2.21\ \sin\left(\frac{2\pi t}{365.25}\right) - 6.83\ \cos\left(\frac{2\pi t}{365.25}\right)$$

Related complex value is $\alpha_{e1} = -6.83 - \mathrm{i}2.21$, a number with amplitude 7.18 °C and phase shift −2.828 radians. As shown in Figure 1.17, the averages over the period 1991–2020 give different values, proving the impact of global warming. The annual average moved from 9.45 to 10.93 °C.

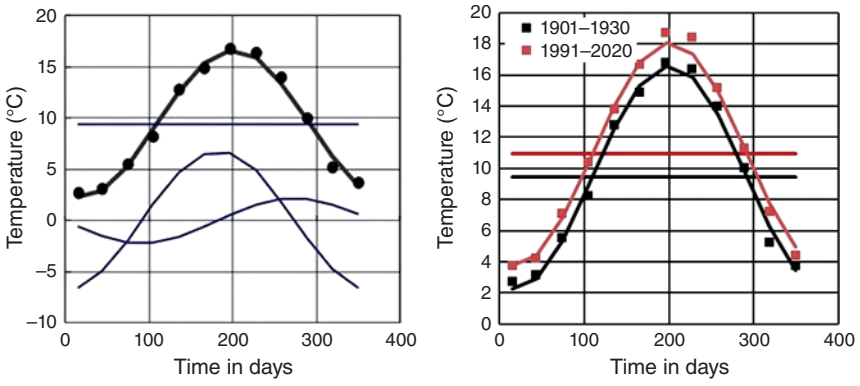


Figure 1.17 Uccle, monthly mean temperature, left: for the period 1901–1930. The black circles are the averages, the thick line gives the result of a Fourier analysis with one harmonic. The thin lines show the annual mean, the sine and the cosine term; right: result for the period 1991–2020. Global warming plays.

Taken is a material layer now, of which one end face endures a periodic temperature and heat flux change from time zero on. The response will be twofold, first a transient that slowly dies and secondly a lasting periodic change. As the layer can neither compress nor extend thermal signals, the periodic passing across must contain the same harmonics as the signal on the end face but the temperature and heat flux amplitudes will dampen and gradually run behind with as result an increasing phase shift during the traverse. Or, the temperature and heat flux in the layer will behave as being complex. The solution so must be a Fourier series with the thermal resistance as independent variable:

$$\theta(R,t) = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha_n(R)\,\exp\left(\frac{2\mathrm{in}\pi t}{T}\right)\right]$$

$$q(t) = -\frac{\mathrm{d}\theta(R,t)}{\mathrm{d}R} = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left[\alpha'_{\mathrm{sn}}(R)\,\exp\left(\frac{2\mathrm{in}\pi t}{T}\right)\right]$$

The accent on α'_{sn} reminds that, mathematically, the complex heat flux is the first derivative of the complex temperature to the thermal resistance, showing as amplitude and phase shift:

Amplitude	Phase shift
$\sqrt{B'^2_{\mathrm{sn}} + A'^2_{\mathrm{sn}}}$	$\mathrm{bgtg}\left(-A'_{\mathrm{sn}}/B'_{\mathrm{sn}}\right)$

1.2.5.2.2 Single-layered

Inserting the equation for the complex temperature into Fourier's second law gives:

$$\sum_{n=-\infty}^{\infty}\left\{\left[\frac{\mathrm{d}^2\alpha_n(R)}{\mathrm{d}R^2} - \frac{2\rho c\lambda\mathrm{in}\pi}{T}\,\alpha_n(R)\right]\exp\left(\frac{2\mathrm{in}\pi t}{T}\right)\right\} = 0 \qquad (1.44)$$

A sum zero requires all coefficients of the time exponentials being zero, or:

$$\frac{\mathrm{d}^2\alpha_n(R)}{\mathrm{d}R^2} - \frac{2\rho c\lambda \mathrm{in}\pi}{T}\alpha_n(R) = 0 \tag{1.45}$$

The outcome so is $2\infty + 1$ second-order differential equations with: the complex temperature as dependent variable and the integer n moving from $-\infty$ over 0 to $+\infty$. As the solutions for n positive and n negative mirror each other, only $\infty + 1$ equations must be solved. All have as solution:

$$\alpha_n(R) = \mathrm{C}_1 \exp(\omega_n R) + \mathrm{C}_2 \exp(-\omega_n R),\ 0 \le n \le \infty \text{ and } \omega_n^2 = \frac{2\rho c\lambda \mathrm{in}\pi}{T} \tag{1.46}$$

The term ω is called the thermal pulsation. Or, Euler's formulas give as complex temperatures and complex heat fluxes in a flat single-layer assembly:

$$\begin{aligned}\alpha_n(R) &= (\mathrm{C}_1 - \mathrm{C}_2)\sin\mathrm{h}(\omega_n R) + (\mathrm{C}_1 + \mathrm{C}_2)\cos\mathrm{h}(\omega_n R)\\ \alpha'_{\mathrm{n}}(R) &= \frac{\mathrm{d}\alpha}{\mathrm{d}R} = \omega_n[(\mathrm{C}_1 - \mathrm{C}_2)\cos\mathrm{h}(\omega_n R) + (\mathrm{C}_1 + \mathrm{C}_2)\sin\mathrm{h}(\omega_n R)]\end{aligned} \tag{1.47}$$

The integration constants $(\mathrm{C}_1 - \mathrm{C}_2)$ and $(\mathrm{C}_1 + \mathrm{C}_2)$ follow from the boundary conditions. These state that the complex temperature $\alpha_{\mathrm{sn}}(0)$ of and complex heat flux $\alpha'_{\mathrm{sn}}(0)$ on end face $R = 0$ must equal:

$$\begin{aligned}\alpha_{\mathrm{sn}}(0) &= (\mathrm{C}_1 - \mathrm{C}_2)\,0 + (\mathrm{C}_1 + \mathrm{C}_2) = \mathrm{C}_1 + \mathrm{C}_2\\ \alpha'_{\mathrm{sn}}(0) &= \omega_n[(\mathrm{C}_1 - \mathrm{C}_2) + (\mathrm{C}_1 + \mathrm{C}_2)\,0] = \omega_n(\mathrm{C}_1 - \mathrm{C}_2)\end{aligned} \tag{1.48}$$

This converts the temperature and heat flux equations into:

$$\begin{aligned}\alpha_n(R) &= \alpha_{\mathrm{sn}}(0)\ \cos\mathrm{h}(\omega_n R) + \alpha'_{\mathrm{sn}}(0)\frac{\sin\mathrm{h}(\omega_n R)}{\omega_n}\\ \alpha'_{\mathrm{n}}(R) &= \alpha_{\mathrm{sn}}(0)\,\omega_n \sin\mathrm{h}(\omega_n R) + \alpha'_{\mathrm{sn}}(0)\ \cos\mathrm{h}(\omega_n R)\end{aligned}$$

Two equations mean two unknowns, whose inclusion in their Fourier series gives the time dependency. Of interest is the link between the complex values on both end faces. Is for envelopes $R = 0$ the inner and $R = R_{\mathrm{T}}$ the outer end face, for partitions R_{T} is the inner end face at the other side. Anyhow, for $R = R_{\mathrm{T}}$ the system becomes:

$$[A_{\mathrm{sn}}(R_{\mathrm{T}})] = [W_n][A_{\mathrm{sn}}(0)] \tag{1.49}$$

with $[A_{\mathrm{sn}}(0)]$ a column matrix of the unknown complex temperature and heat flux on the inner end face, $[A_{\mathrm{sn}}(R_{\mathrm{T}})]$ a column matrix of the complex temperature and heat flux on the other end face and $[W_n]$ the system matrix of the nth harmonic, depending on the assembly's thickness, its material properties, the base period and the harmonic considered. $[W_n]$ so replaces the thermal resistance, although containing additional information.

$\boldsymbol{n} = 0$ Then, $\alpha_{\mathrm{so}}/2$ and $\alpha'_{\mathrm{so}}/2$ represent the average temperature and heat flux on the inner end face, while the thermal pulsation then becomes:

$$\omega_0^2(n = 0) = 2\mathrm{i}\rho c\lambda 0\pi/T = 0$$

This makes the complex surface temperature and heat flux on the other end face equal to:

$$\alpha_{\mathrm{so}}(R_{\mathrm{T}}) = \alpha_{\mathrm{so}}(0) + \alpha'_{\mathrm{so}}(0)\frac{0}{0} \quad \alpha'_{\mathrm{so}}(R_{\mathrm{T}}) = \alpha_{\mathrm{so}}(0)\,0 + \alpha'_{\mathrm{so}}(0) = \alpha'_{\mathrm{so}}(0)$$

Or, the in- and outgoing average heat fluxes then are the same. As this holds on each plan-parallel plane between 0 and R_T in the single-layer assembly, $n = 0$ means steady state. The ratio 0/0 in the temperature equation can be clarified using l' Hospital's rule:

$$\lim_{\omega_0 \to 0} [\sin h(\omega_o R_T)/\omega_o] = \lim_{\omega_0 \to 0} [R_T \cos h(\omega_o R_T)] = R_T$$

which gives:

$$\alpha_{so}(R_T) = \alpha_{so}(0) + \alpha'_{so}(0)\, R_T$$

a result, holding all over the assembly. The average temperature curve in the $[R,\theta]$-axis system so is a straight line with the heat flux as slope. This extends the concept steady state to the average over a long enough period of time.

$\boldsymbol{n > 0}$ A first case is the temperature on the inner end face remaining constant. The complex temperature on the other end face then becomes:

$$\alpha_{sn}(R_T) = \alpha'_{sn}(0) \frac{\sin h(\omega_n R_T)}{\omega_n} \quad \text{or} \quad \frac{\alpha_{sn}(R_T)}{\alpha'_{sn}(0)} = \frac{\sin h(\omega_n R_T)}{\omega_n} \tag{1.50}$$

The function on the right now stands for the ratio between the complex surface temperature on the other and the complex heat flux on the inner face. In steady state, this ratio is the thermal resistance. Therefore, that function is called the dynamic thermal resistance for the nth harmonic, symbol D_q^n, units $m^2{\cdot}K/W$, with as amplitude and time shift $\left(\varphi_q^n\right)$:

$$\left[D_q^n\right] = [\sin h(\omega_n R_T)/\omega_n] \qquad \phi_\theta^n = \arg[\sin h(\omega_n R_T)/\omega_n]$$

The assumption made looks theory. However, the case refers to spaces at constant temperature. For the period infinitely long, the amplitude and phase shift of the dynamic thermal resistance become:

$$\left[D_q^n\right] = \lim_{n \to 0} \left[\frac{\sin h(0)}{0}\right] = \lim_{n \to 0} [R_T \cos h(0)] = R_T \qquad \phi_q^n = \lim_{n \to 0} \left[\arg \frac{\sin h(0)}{0}\right] = \infty$$

Or, its value equals the steady state one then. Logic as infinitely long means steady state. For a really fast oscillation with period ≈0 s, the amplitude and phase shift of the dynamic thermal resistance change to:

$$\left[D_q^n\right] = \lim_{n \to \infty} \left[\frac{\sin h(\infty)}{\infty}\right] = \lim_{n \to \infty} [R_T \cos h(\infty)] = \infty \qquad \phi_q^n = \lim_{n \to \infty} \left[\arg \frac{\sin h(\infty)}{\infty}\right] = 0$$

Or, ever faster oscillations push its value to infinite, meaning the assembly dampens the signal completely. The dynamic thermal resistance so is always larger than the steady state one, meaning that imposing high thermal resistances suffices to get a high dynamic value.

A second case is the heat flux on the inner end face constant. As no complex heat fluxes exist there then $\left(a'_{sn}(0) = 0\right)$, the complex temperature on the other end face becomes:

$$\alpha_{sn}(R_T) = \alpha_{sn}(0)\, \cos h(\omega_n R_T) \quad \text{or} : \quad \frac{\alpha_{sn} R_T}{\alpha_{sn}(0)} = \cos h(\omega_n R_T) \tag{1.51}$$

The function on the right so stands for the ratio between the complex temperatures on the other and the inner end face. It is called the temperature damping for the *n*th harmonic, symbol D_θ^n with as amplitude and time shift $\left(\varphi_\theta^n\right)$:

$$\left[D_q^n\right] = [\cos h(\omega_n R_T)] \quad \phi_\theta^n = \arg[\cos h(\omega_n R_T)]$$

At first glance, that quantity looks fictional, the more because there's no steady state equivalent. Anyhow, it shows the ability of an assembly to moderate the impact at the inner end face of a temperature change at the other. In cases with few heat gains indoors and a restricted ventilation, an enclosure with high temperature damping might so keep the indoor temperature quite stable. This gives it a performance status. For the period infinitely long, the amplitude and phase of the temperature damping become:

$$\left[D_\theta^n\right] = \lim_{n\to 0}[\cos h(0)] = 1 \quad \phi_\theta^n = \lim_{n\to 0}[\arg[\cos h(0)]] = \infty$$

Or, it then nears 1. In fact, in steady state, without heat gains and losses, the temperatures at both end faces must remain the same. For a period nearing 0 s, the amplitude and phase of the temperature damping become:

$$\left[D_\theta^n\right] = \lim_{n\to \infty}[\cos h(\infty)] = \infty \quad \phi_\theta^n = \lim_{n\to \infty}[\arg[\cos h(\infty)]] = 0$$

Or, ever faster fluctuations push its value to infinite.

A third case is the temperature on the outer or the end face at the other side constant. The complex temperature of the inner end face then converts to:

$$0 = D_\theta^n\, \alpha_{sn}(0) + D_q^n\, \alpha'_{sn}(0) \;\text{ or }:\; \frac{\alpha'_{sn}(0)}{\alpha_{sn}(0)} = -\frac{D_\theta^n}{D_q^n} = -\omega_n \cot\, gh(\omega_n R) \qquad (1.52)$$

The function on the right, equal to the ratio between the complex heat flux and temperature on the inner end face, thus to the ratio between the temperature damping $\left(D_\theta^n\right)$ and the dynamic thermal resistance $\left(D_q^n\right)$, is called the admittance for the *n*th harmonic, symbol Ad^n, units W/(m²·K), with as amplitude and time shift $\left(\varphi_{Ad}^n\right)$:

$$[Ad^n] = \left[-\omega_n \cot\, gh(\omega_n R_T)\right] \qquad \varphi_{Ad}^n = \varphi_\theta^n - \varphi_q^n$$

Again, the quantity looks fictional. However, it indicates how easily single-layer assemblies pick up heat when the temperature on the inner end face fluctuates. The higher its amplitude, the more effective heat is stored. Or, the assessment 'capacitive' points to envelope assemblies or partitions having a high admittance. The thermal pulsation ω_n now also writes as:

$$\omega_n = \sqrt{i}\,\sqrt{\rho c \lambda}\,\sqrt{2n\pi/T}$$

This shows that a high admittance requires a high value of the square root of the product of the volumetric heat capacity with the thermal conductivity, a quantity called the contact coefficient or effusivity of the material, symbol *b*, units J/(m²·s$^{-1/2}$·K). The higher its effusivity, take stony materials compared to timber, the more active the material as heat storage medium. For a period infinitely long, the amplitude and phase of the admittance become:

$$[Ad^n] = \lim_{n\to 0}\left(\frac{D_\theta^n}{D_q^n}\right) = \frac{1}{R_T} \qquad \phi_{As}^n = \lim_{n\to 0}\left[\arg\left(-\frac{D_\theta^n}{D_q^n}\right)\right] = \infty$$

Or, ever slower fluctuations push the admittance to the thermal conductance (P). Logic, as in steady state, the heat flux entering or leaving an assembly equals $P\Delta\theta_s$. For a period nearing 0 s, the amplitude and phase of the admittance touch:

$$[\mathrm{Ad}^n] = \lim_{n\to\infty}\left(\frac{\mathrm{D}_\theta^n}{\mathrm{D}_q^n}\right) = \infty \qquad \phi_{\mathrm{As}}^n = \lim_{n\to\infty}\left[\arg\left(\frac{\mathrm{D}_\theta^n}{\mathrm{D}_q^n}\right)\right] = 0$$

Or, ever faster fluctuations push the admittance to infinite.

With the dynamic thermal resistance and the temperature damping known, the system matrix $[W_n]$, see above, can be rewritten as:

$$[W_n] = \begin{bmatrix} \mathrm{D}_\theta^n & \mathrm{D}_q^n \\ \omega_n^2\,\mathrm{D}_q^n & \mathrm{D}_\theta^n \end{bmatrix}$$

A transposition to real numbers first requires reformulating the thermal pulsation:

$$\omega_n = \sqrt{i}\,b\sqrt{\frac{2n\pi}{T}} = \frac{(1+i)\,b\sqrt{\frac{2n\pi}{T}}}{\sqrt{2}} = (1+i)\,b\sqrt{\frac{n\pi}{T}}$$

The conversion of $\sqrt{i}$ is based on $(1+i)^2 = 2i$ or $\sqrt{i} = (1+i)/\sqrt{2}$. The product $\omega_n R$ so becomes:

$$\omega_n R = \frac{(1+i)\,b\cdot\sqrt{\frac{n\pi}{T}}}{\lambda} = (1+i)\cdot\sqrt{\frac{n\pi}{aT}}$$

with x stretching along the single-layer assembly's thickness and $a = \lambda^2/b^2$ its thermal diffusivity. Setting $X_n = x\sqrt{n\pi/aT}$ allows rewriting the thermal pulsation as $(1+i)X_n/R$, giving for the temperature damping D_θ^n:

$$\mathrm{D}_\theta^n = \cos\mathrm{h}(\omega_n R) = \cos\mathrm{h}[(1+i)X_n] = \cos\mathrm{h}(X_n)\cos\mathrm{h}(iX_n) + \sin\mathrm{h}(X_n)\sin\mathrm{h}(iX_n)$$

or, with $\cos\mathrm{h}(iX_n) = \cos(X_n)$ and $\sin\mathrm{h}(iX_n) = i\sin(X_n)$:

$$\cos\mathrm{h}(\omega_n R) = \cos\mathrm{h}(X_n)\ \cos(X_n) + i\ \sin\mathrm{h}(X_n)\ \sin(X_n)$$

Analogously, the dynamic thermal resistance D_q^n and admittance Ad^n of the single-layer assembly turn into:

$$\mathrm{D}_q^n = \frac{\sin\mathrm{h}(\omega_n R)}{\omega_n} = \frac{R}{2\mathrm{X}_n}\left\{\begin{array}{l}[\sin\mathrm{h}(X_n)\cos(X_n) + \cos\mathrm{h}(X_n)\sin(X_n)] \\ \quad +i\,[\cos\mathrm{h}(X_n)\sin(X_n) - \sin\mathrm{h}(X_n)\cos(X_n)]\end{array}\right\}$$

$$\mathrm{Ad}^n = \omega_n\sin\mathrm{h}(\omega_n R) = \frac{X_n}{R}\left\{\begin{array}{l}[\sin\mathrm{h}(X_n)\cos(X_n) - \cos\mathrm{h}(X_n)\sin(X_n)] \\ \quad +i\,[\cos\mathrm{h}(X_n)\sin(X_n) + \sin\mathrm{h}(X_n)\cos(X_n)]\end{array}\right\}$$

The temperature damping, the dynamic thermal resistance and the term $\omega_n^2\mathrm{D}_q^n$ in the system matrix can be rewritten as:

$$\mathrm{D}_\theta^n = \cos\mathrm{h}(\omega_n R) = \mathrm{G}_{n1} + i\mathrm{G}_{n2}$$

$$\mathrm{D}_q^n = \frac{\sin\mathrm{h}(\omega_n R)}{\omega_n} = R\,(\mathrm{G}_{n3} + i\mathrm{G}_{n4})$$

$$\omega_n^2\mathrm{D}_q^n = \omega_n\sin\mathrm{h}(\omega_n R) = \frac{\mathrm{G}_{n5} + i\mathrm{G}_{n6}}{R}$$

The G-functions in these three formulas equal:

$$G_{n1} = \cos h(X_n)\cos(X_n) \quad G_{n2} = \sin h(X_n)\sin(X_n)$$

$$G_{n3} = [\sin h(X_n)\cos(X_n) + \cos h(X_n)\sin(X_n)]/(2X_n)$$

$$G_{n4} = [\cos h(X_n)\sin(X_n) - \sin h(X_n)\cos(X_n)]/(2X_n)$$

$$G_{n5} = X_n[\sin h(X_n)\cos(X_n) - \cos h(X_n)\sin(X_n)]$$

$$G_{n6} = X_n[\cos h(X_n)\sin(X_n) + \sin h(X_n)\cos(X_n)]$$

The amplitude and phase shift of the temperature damping, the dynamic thermal resistance and the admittance so become:

Amplitude	Phase shift
$\left[D_q^n\right] = R_T\sqrt{G_{n3}^2 + G_{n4}^2}$	$\varphi_q^n = \text{bgtg}(G_{n4}/G_{n3})$
$\left[D_\theta^n\right] = \sqrt{G_{n1}^2 + G_{n2}^2}$	$\varphi_q^n = \text{bgtg}(G_{n2}/G_{n1})$
$[Ad^n] = \left[D_\theta^n\right] / \left[D_q^n\right]$	$\varphi_{Ad}^n = \varphi_\theta^n - \varphi_q^n$

With all this, per harmonic n the complex 2×2 matrix converts into following real 4×4 matrix:

$$W_n = \begin{bmatrix} G_{n1} & G_{n2} & R_T G_{n3} & R_T G_{n4} \\ -G_{n2} & G_{n1} & -R_T G_{n4} & R_T G_{n3} \\ G_{n5}/R_T & G_{n6}/R_T & G_{n1} & G_{n2} \\ -G_{n6}/R_T & G_{n5}/R_T & -G_{n2} & G_{n1} \end{bmatrix}$$

What concerns the phase shifts, Figure 1.18 shows which quarter convenes with which sign of the G-functions.

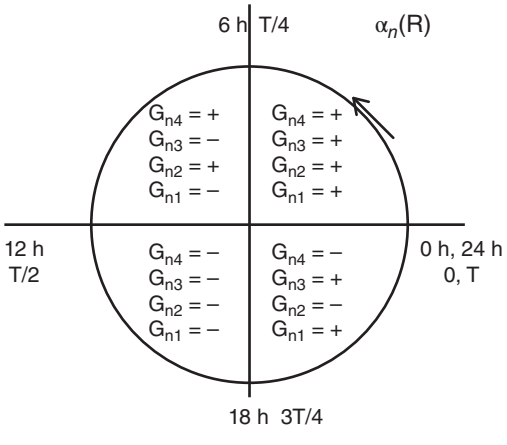


Figure 1.18 Phase shift: sign of the G-functions.

1.2.5.2.3 Multi-layered

Multi-layer assemblies behave as a serial composition of flat layers, of which each has its own layer matrix $[W_{n,i}]$. The system matrix $[W_{nT}]$ of the whole then becomes:

$$[\mathrm{A_{sn}}(R_\mathrm{T})] = W_{nT}[\mathrm{A_{sn}}(0)] \tag{1.53}$$

with $[A_{\mathrm{sn}}(R_\mathrm{T})]$ and $[A_{\mathrm{sn}}(0)]$ the column matrixes of the nth complex temperature and heat flux on both end faces, see Figure 1.19.

The relation between the complex temperatures and heat fluxes at the interface j between layer j and the one adjacent layer and $j+1$ between layer j and the other adjacent layer is:

$$[\mathrm{A}_{n,j+1}] = W_{n,j}[\mathrm{A}_{nj}]$$

For the inner end face coinciding with R = 0, the inner end face, the series of layer matrices for an assembly counting m layers becomes:

$$\begin{aligned}
[A_{n,1}] &= W_{n,1}[A_{\mathrm{s,n}}(0)] \\
[A_{n,2}] &= W_{n,2}[A_{n,1}] \\
&\cdots \\
[A_{n,m-1}] &= W_{n,m-1}[A_{n,m-2}] \\
[A_{\mathrm{s},n}(R_\mathrm{T})] &= W_{n,m}[A_{n,m-1}]
\end{aligned}$$

Transposing now each preceding equality in the next gives:

$$[A_{\mathrm{s},n}(R_\mathrm{T})] = W_{n,m}W_{n,m-1}\ldots W_{n,2}W_{n,1}[A_{\mathrm{s},n}(0)]$$

Or, the system matrix of a multi-layer assembly equals:

$$[W_{nT}] = \prod_{j=1}^{m}[W_{n,j}] \tag{1.54}$$

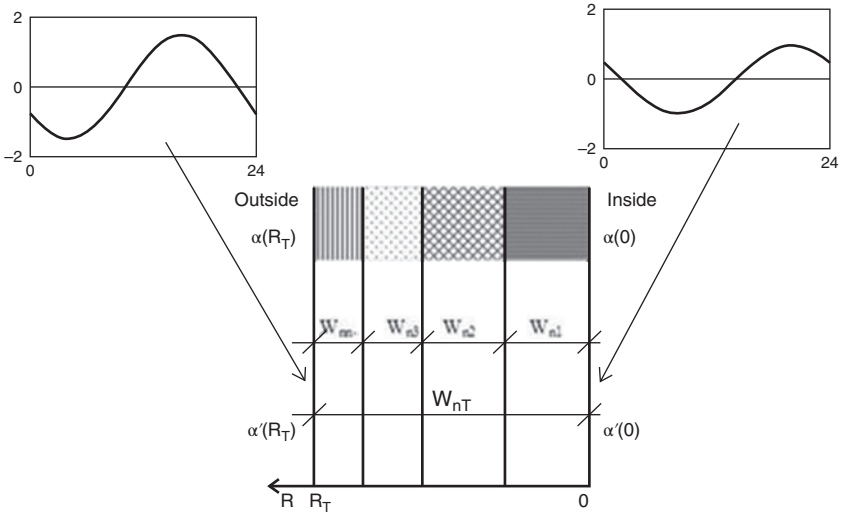


Figure 1.19 Composite assembly, system matrix.

Multiplying each next layer matrix with the product of all preceding layer matrices starts with the layer at the inner end face to end with the layer at the other end face. As the commutation property does not apply for a product of matrices, the layer sequence matters. Or, contrary to the thermal resistance, the transient response of a multi-layer assembly changes with how that sequence looks! Of course, each layer matrix keeps the same formulation as for a single-layer assembly. If two of the complex temperatures and heat fluxes at the end faces are known, then the other two follow from:

$$[A_{\mathrm{sn}}(R_{\mathrm{T}})] = [W_{n\mathrm{T}}][A_{\mathrm{sn}}(0)]$$

To quantify the complex temperatures of and heat fluxes at all interfaces, the layer equations must be ascended or descended in the correct order.

Besides, to get the complex temperature and heat flux course in a single-layer assembly, it should be divided in m layers of thickness Δx and layer matrix $[W_{\Delta x}]$, giving as system matrix $[W_n] = (W_{\Delta x})^m$. Further calculations then are as for multi-layer.

1.2.5.3 Any Boundary Conditions, Flat Assemblies

1.2.5.3.1 Convolution

Convoluting starts from Dirac impulses (Figure 1.20), whereby the temperature or heat flux in a single- or multi-layer assembly remains 0, unless for an infinitely short time step dt, when at one of the end faces it jumps to 1.

If this is the temperature θ_{s1} on end face s1, then after a time also the heat flux $q_{\mathrm{s1}}(t)$ there and the temperature $\theta_{\mathrm{s2}}(t)$ and heat flux $q_{\mathrm{s2}}(t)$ on the other end face s2 will vary a little. In case the materials used have constant properties, these changes $q_{\mathrm{s1}}(t)$, $\theta_{\mathrm{s2}}(t)$ and $q_{\mathrm{s2}}(t)$, called 'response factors', are written as:

$$I_{\theta_{\mathrm{s1}}\mathrm{q}_{\mathrm{s1}}} \quad I_{\theta_{\mathrm{s1}}\theta_{\mathrm{s2}}} \quad I_{\theta_{\mathrm{s1}}\mathrm{q}_{\mathrm{s2}}}$$

Analogously, for the heat flux on end face s1 jumping to 1 or the temperature or heat flux on end face s2 jumping to 1, related response factors look:

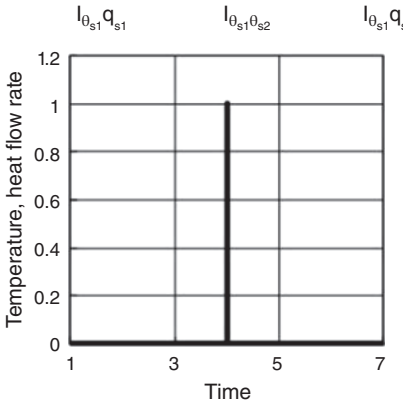


Figure 1.20 Dirac impulse.

Heat flux, end face 1	$I_{q_{s1}\theta_{s1}}$	$I_{q_{s1}\theta_{s2}}$	$I_{q_{s1}q_{s2}}$
Temp., end face 2	$I_{\theta_{s2}q_{s1}}$	$I_{\theta_{s2}\theta_{s1}}$	$I_{\theta_{s1}q_{s2}}$
Heat flux, end face 2	$I_{q_{s2}q_{s1}}$	$I_{q_{s2}q_{s1}}$	$I_{q_{s2}q_{s2}}$

For multi-layer assemblies, the response factors depend on the layer sequence. As a consequence, their values change with on which end face both jump: s1 or s2.

If the temperature or heat flux on end face s1 shows a jump with value θ_o or q_o, then, with the response factors known, the temperature and heat flux change on end face s2 and the heat flux or temperature on end face s1 become:

Pulse θ_o on end face s1	**Pulse q_o on end face s1**
$q_{s1} = \theta_o I_{\theta_{s1}q_{s1}}$	$\theta_{s1} = q_o I_{q_{s1}\theta_{s1}}$
$\theta_{s2} = \theta_o I_{\theta_{s1}\theta_{s2}}$	$\theta_{s2} = q_o I_{q_{s1}\theta_{s2}}$
$q_{s2} = \theta_o I_{\theta_{s1}q_{s2}}$	$q_{s2} = q_o I_{q_{s1}q_{s2}}$

A jump on end face s2 gives analogous relations. Any signal $\theta_{s1}(t)$ on end face s1 can be seen now as an addition of jumps $\theta_{s1}(t)\Delta t$, giving as temperature (θ_{s2}) on end face s2 (Figure 1.21):

$$t = 0 \quad \theta_{s2}(0) = 0$$

$$t = \Delta t \quad \theta_{s2}(\Delta t) = \theta_{s1}(t = 0)\, I_{\theta_{s1}\theta_{s2}}(t = \Delta t)$$

$$t = 2\Delta t \quad \theta_{s2}(2\Delta t) = \theta_{s1}(t = 0)\, I_{\theta_{s1}\theta_{s2}}(t = 2\Delta t) + \theta_{s1}(t = \Delta t)\, I_{\theta_{s1}\theta_{s2}}(t = \Delta t)$$

$$t = 3\Delta t \quad \theta_{s2}(3\Delta t) = \theta_{s1}(t = 0)\, I_{\theta_{s1}\theta_{s2}}(t = 3\Delta t) + \theta_{s1}(t = \Delta t)\, I_{\theta_{s1}\theta_{s2}}(t = 2\Delta t) + \theta_{s1}(t = 2\Delta t) I_{\theta_{s1}\theta_{s2}}(t = \Delta t)$$

…

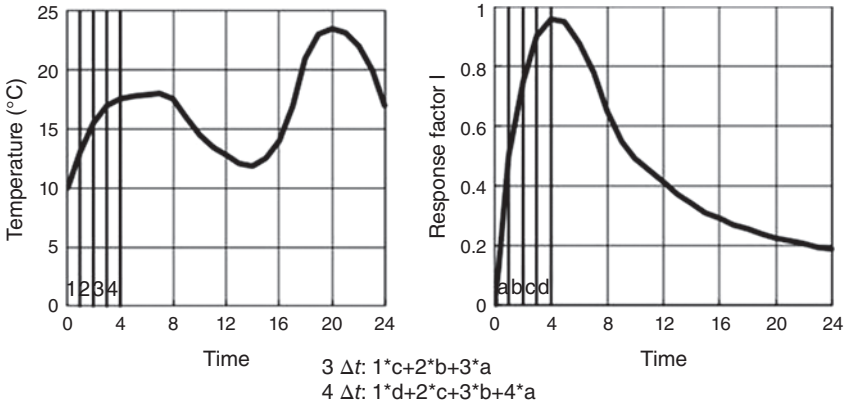


Figure 1.21 Convolution, principle.

$$t = n\Delta t \quad \theta_{s2}(n\Delta t) = \theta_{s1}(t=0)\, I_{\theta_{s1}\theta_{s2}}(t = n\Delta t) + \theta_{s1}(t = \Delta t) I_{\theta_{s1}\theta_{s2}}(t = (n-1)\Delta t) + \theta_{s1}(t = 2\Delta t) I_{\theta_{s1}\theta_{s2}}(t = (n-2)\Delta t) + \cdots$$

$$\text{or} : \theta_{s2}(n\Delta t) = \sum_{j=0}^{n-1} \theta_{s1}(j\Delta\vec{t})\, I_{\theta_{s1}\theta_{s2}}((n-j)\Delta t)$$

Written as an integral with the time steps infinitely small:

$$\theta_{s2}(t) = \int_0^t \theta_{s1}(\tau)\, I_{\theta_{s1}\theta_{s2}}(t-\tau)\mathrm{d}\tau$$

Defined so is the convolution integral of the temperature θ_{s2} on end face s2 for a signal $\theta_{s1}(t)$ on end face s1. The signal $\theta_{si}(\tau)$ has to be scanned clockwise, and the response factor $I_{\theta_{s1}\theta_{s2}}(t-\tau)$ counterclockwise. A same approach, be it with the correct response factors, holds for the heat fluxes on both end faces, more, for all situations with randomly changing end face conditions. Response factors and convolution integrals can only be calculated numerically.

1.2.5.3.2 Temperature Change in a Semi-infinitely Thick Layer

At time 0, the surface temperature of a uniformly warm, semi-infinitely thick layer suddenly jumps from θ_{so} to $\theta_{so} + \Delta\theta_{so}$ (Figure 1.22). Solving Fourier's second law, in this case, requires a separation of variables with as initial condition $t = 0$, $\theta = \theta_{s,o}$ for $0 \le x \le \infty$ and as boundary condition on the surface: $t \ge 0$, $\theta_s = \theta_{so} + \Delta\theta_{so}$.

The solution is:

$$\theta(x,t) = \theta_{s0} + \Delta\theta_{s0}\left(\frac{2}{\sqrt{\pi}}\int_{q=\frac{x}{2\sqrt{at}}}^{\infty} \exp(-q^2)\,\mathrm{d}q\right) \tag{1.55}$$

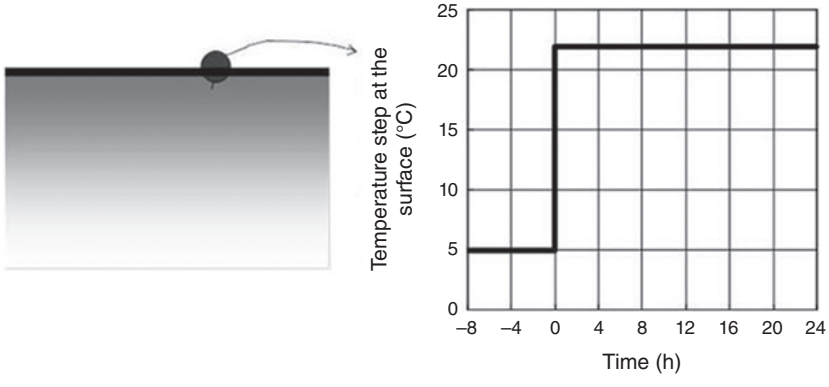


Figure 1.22 Temperature jump at the surface of a uniformly warm semi-infinitely thick layer.

The term between brackets stands for the inverse error function with '*a*' the thermal diffusivity of the medium. The heat flux ($x = 0$) on the surface in turn equals:

$$q = -\lambda \frac{\mathrm{d}\theta}{\mathrm{d}x} = -\lambda \left[\frac{2\Delta\theta_{s0}}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at} \right) \right]_{x=0} = -\frac{\Delta\theta_{s0}\, b}{\sqrt{\pi t}} \tag{1.56}$$

with b the contact coefficient of the medium. Applying the definition of response factor to this equation gives:

$$\mathrm{I}_{\theta_{s1} q_{s1}} = -b/\sqrt{\pi\, t}$$

Semi-infinite mediums do not exist, but the soil and really thick material layers are close. In any case, the heat flux equation illustrates what the contact coefficient is doing. A high value means fast heating with increasing and fast cooling with decreasing surface temperature, and a low value both going slowly. Materials with high contact coefficient so act as storage volumes. If in passive solar buildings walls and floors cannot temporarily store a large part of the solar gains, the indoors could become far too warm. Or, partitions and floors in such buildings should consist of heavy materials and be at least as thick as structurally needed. The heat in- or outflow per m^2 from time 0 on in such semi-infinite medium equals:

$$Q = \int_0^t q\, \mathrm{d}t = \frac{2b\, \Delta\theta_{s0}}{\sqrt{\pi}} \sqrt{t} = A_q \sqrt{t} \tag{1.57}$$

with A_q the heat absorption coefficient, units $J/(m^2 \cdot K \cdot s^{1/2})$.

In case two materials, one at temperature θ_1, the other at temperature θ_2, contact each other, the heat flux there will equal ($\theta_1 > \theta_2$):

$$\text{In material 1} : q_{s1} = (\theta_1 - \theta_c) b_1 / \sqrt{\pi t}$$

$$\text{In material 2} : q_{s2} = (\theta_c - \theta_2) b_2 / \sqrt{\pi t}$$

As both fluxes must have the same absolute value, the contact temperature θ_c will be:

$$\theta_c = \frac{b_1 \theta_1 + b_2 \theta_2}{b_1 + b_2} \tag{1.58}$$

Its value so depends on the temperature and contact coefficient of the materials contacting each other, a reality showing what people experience when touching materials. Those with high contact coefficient feel cold, and those with low contact coefficient comfortably warm. Indeed, in the first case, the skin temperature, 32–33°C, will move direction surface temperature of the material, while in the second case, the surface temperature of the material will near the skin temperature. Concrete and aluminium so feel unpleasant, wood pleasant. The term cold and warm materials is often used. Chair finishes should so best have a low contact coefficient.

1.2.5.4 Two and Three Dimensions: Thermal Bridges

To quantify heat transmitted two or three dimensionally, Fourier's second law for transient conduction in 2 or 3 dimensions, applies. Since analytical solutions fail, CVM is used. The principles were explained. Now, additionally, the heat capacity $\rho c \Delta V$ of each control volume plays. Without dissipation, the resulting heat flow per time step from any adjacent control volume to the central equals the change in heat during that step stored in or released by the central one:

$$\left(\sum \Phi_m\right) \Delta t = \rho c \, \Delta V \, \Delta\theta$$

In it, Φ_m is the weighted average heat flow during time Δt between adjacent and central control volume:

$$\Phi_m = p\Phi_{t+\Delta t} + (1-p)\,\Phi_t \quad (1 > p > 0)$$

The balance equation so rewrites as ($i = l$, m, n and $j = \pm 1$):

$$p\left[\sum (P_{s,i+j}\theta_{i+j}) - \theta_{l,m,n} \sum P_{s,i+j}\right]^{(2)} + (1-p)$$
$$\cdot \left[\sum (P_{s,i+j}\theta_{i+j}) - \theta_{l,m,n} \sum P_{s,i+j}\right]^{(1)} = \rho c \, \Delta V \frac{\theta^{(2)}_{l,m,n} - \theta^{(1)}_{l,m,n}}{\Delta t}$$

In it, $p = 0$ means forward, $p = 1$ backward and $p = 0.5$ average differencing, called the Cranck–Nicholson scheme that after rearrangement looks:

$$\left[\sum (P_{s,i+j}\theta_{i+j}) - \theta_{l,m,n}\left(\sum P_{s,i+j} + \frac{2\rho c \, \Delta V}{\Delta t}\right)\right]^{(2)}$$
$$= \left[-\sum (P_{s,i+j}\theta_{i+j}) + \theta_{l,m,n}\left(\sum P_{s,i+j} - \frac{2\rho c \, \Delta V}{\Delta t}\right)\right]^{(1)}$$

In it, the actual temperatures (2) are the unknowns, the preceding (1) the known ones. Each time step so gives a system with as many equations as control volumes with unknown temperature. With the initial and boundary conditions known, the system so found is solved time step wise. A question of course is: which meshes and time steps to use? Best choice is to refine the meshes in materials with high thermal diffusivity and to shorten the time step where mesh density is higher, the faster the boundary conditions change and/or the higher the desired accuracy.

1.3 Heat Exchange at Surfaces by Convection and Radiation

1.3.1 What?

Till now, the temperatures on the end faces of an assembly were assumed to be known. However, in most cases, it's the air temperature at both sides, sometimes the heat flux at a surface, that is known. In fact, weather stations measure the air temperature outdoors, while logging the temperature in the air indoors is simpler than logging surface temperatures. Therefore, the focus now moves to the heat transfer ambient to ambient through assemblies. Of course, the key is to know how heat

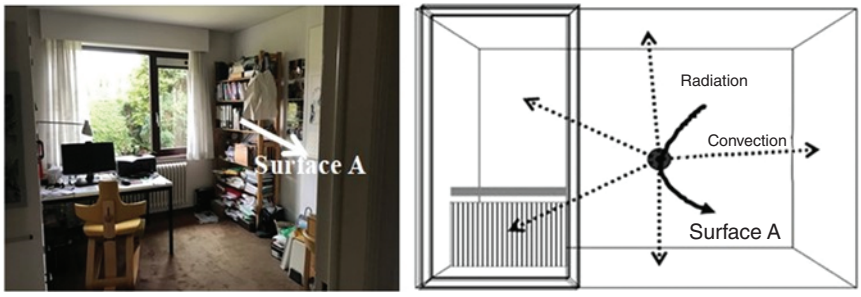


Figure 1.23 Heat exchange at surface *A* by convective exchange with the room air and radiation to and from all other surfaces.

from the ambient reaches a surface. Two modes intervene: convection between the air and the surface called A in Figure 1.23 and radiant exchanges with all faces that surface sees.

1.3.2 Convection

1.3.2.1 In General

Here, the term 'convection' refers to the heat exchanged between a liquid or a gas and a surface they are contacting. Related heat flux and flow are commonly written as:

$$q_c = h_c(\theta_{fl} - \theta_s) \quad \Phi_c = h_c(\theta_{fl} - \theta_s)\,A \tag{1.59}$$

with θ_{fl}, the temperature in the said liquid or gas close to the surface, θ_s the surface temperature and h_c a property called the convective surface film coefficient, units W/(m^2·K). Both equations are called 'Newton's law'. It links convection linearly to the driving temperature difference $(\theta_{fl} - \theta_s)$. The law anyhow rather serves as definition of the surface film coefficient, which reflects the complexity of the heat and mass flow in a liquid or gas touching a surface. Are fixing the mass flow: the scalar conservation of mass and vectorial conservation of momentum equations:

$$\text{Mass (scalar)} \quad \text{div}(\rho\mathbf{v}) = 0$$

$$\text{Momentum (Navier–Stokes, vector)} \quad \frac{\mathrm{d}(\rho\mathbf{v})}{\mathrm{d}t} = \rho\mathbf{g} - \mathbf{grad}\,P + \mu\nabla^2\mathbf{v}$$

In both, ρ is density of the fluid, μ its dynamic viscosity, P the total pressure and $\rho\mathbf{g}$ the gravity gradient. Unknowns are the three velocity components v_x, v_y, v_z and the total pressure. When the flow turns turbulent, how to shape turbulency adds. Often used for that is the (k,ε)-approach, with k the turbulent kinetic energy and ε its dissipation. Two additional scalar equations so add, one for k and one for ε. In addition, estimating the three velocity components in the x, y, and z direction turns the vectorial momentum equation in three scalar equations, so that quantifying turbulent flow requires six scalar equations with six unknowns. Constant properties, negligible kinetic energy and hardly any friction in turn allow to write the conservation of energy in a moving fluid with thermal diffusivity a as:

$$a\nabla^2\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{1.60}$$

In the Navier–Stokes and in this equation, d/dt is the total derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial}{\partial z}\frac{\partial z}{\partial t} + \frac{\partial}{\partial t} = \underbrace{\frac{\partial}{\partial x}v_x}_{(1)} + \underbrace{\frac{\partial}{\partial y}v_y}_{(2)} + \underbrace{\frac{\partial}{\partial z}v_z}_{(3)} + \frac{\partial}{\partial t}$$

In the conservation of energy equation, the terms (1), (2) and (3) multiplied with $\rho c\theta$ give the enthalpy, the θ °C warm liquid or gas displace at a velocity with components v_x, v_y, v_z. Temperature θ figures as seventh unknown. All information about convection follows from solving that system of seven scalar partial differential equations equations. With the temperature field known, then, due to convection becoming conduction in the laminar boundary layer against the surface, the heat flux to it follows from Fourier's first law, giving as surface film coefficient:

$$h_\mathrm{c} = -\lambda_\mathrm{fl}\frac{(\mathrm{grad}\,\theta)_\mathrm{s}}{\theta_\mathrm{fl} - \theta_\mathrm{s}} \tag{1.61}$$

In it, $(\mathrm{grad}\,\theta)_\mathrm{s}$ is the temperature gradient in the laminar boundary layer and θ_fl the temperature in the undisturbed liquid or gas. Although Computerized Fluid Dynamics (CFD) is mostly used, solving laminar natural convection along a semi-infinite vertical surface and laminar forced convection along a semi-infinite horizontal surface is analytically doable. In both cases, the velocity and temperature equations look similar, which with air as gas gives:

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_y}{\partial y} = \nu_\mathrm{fl}\frac{\partial^2 v_x}{\partial y^2} \quad v_x\frac{\partial \theta}{\partial x} + v_y\frac{\partial \theta}{\partial y} = a_\mathrm{fl}\frac{\partial^2 \theta}{\partial y^2}$$

For the thermal diffusivity equal to the kinematic viscosity, the three equations even become identical. The so coinciding heat and mass flow patterns give as surface film coefficient:

$$h_{c,x} = \frac{0.664\ \lambda_\mathrm{fl}}{x}\sqrt{\frac{v_\infty x}{\nu_\mathrm{fl}}}$$

with x the distance to the surface's free edge and v_∞ the velocity of the undisturbed air. The result is a convective surface film coefficient that changes along the surface. However, any trial to calculate analytically the convective heat transfer along walls in a room, along the envelope outdoors or along any other surface is doomed to fail. Happily, even in complex situations, convection depends on all parameters and properties fixing the heat and mass flow:

Fluid properties	**Flow parameters**
Thermal conductivity (λ_fl)	Geometry
Density (ρ_fl)	Surface roughness
Specific heat capacity (c_fl)	Temperature difference ($\theta_\mathrm{fl} - \theta_\mathrm{s}$)
Kinematic viscosity ($\mu_\mathrm{fl}/\rho_\mathrm{fl}$)	Nature and direction of the flow
Volumetric expansion coefficient	Velocity components v_x, v_y, v_z

1.3.2.2 Typology

1.3.2.2.1 Driving Forces

In case differences in air density due to temperature and vapour concentration gradients act as driving force, natural convection with a flow pattern mainly reflecting the temperature field in the air is the result, typically the case indoors. If instead an imposed pressure difference acts as driving force, then forced convection is a fact with a flow pattern independent of temperature, usually the case outdoors. Both convection types can coincide, giving mixed convection. Outdoors, a natural component adds when close to windless. Indoors, opening and closing doors may add forced convection peaks to natural convection.

1.3.2.2.2 Flow Types

Mass flows can be laminar, turbulent or transient. If laminar, streamlines never cross and particle velocities and the overall flow velocity are the same. In turbulent flow, a chaotic momentum field creates whirling eddies with particle velocities different from the overall flow velocity. Turbulent kinetic energy (k) builds up in the eddies while turbulent dissipation (ε) continuously dampens their motion. Describing turbulent flow is hard, even with CFD and a fractional eddy approach. Transient flow in turn fills the gap between both. Small disturbances along a flow path may induce switches from laminar to turbulent, after which the eddies formed fade away and the flow turns laminar again. In between, it is transient. Turbulency favours heat transfer the most, because in laminar flow, heat conduction can only develop perpendicular to the flow direction. So:

flow → Driving force↓	Laminar	Transient	Turbulent
Density differences	X	X	X
Density and pressures	X	X	X
Pressures	X	X	X

1.3.2.3 Quantifying the Convective Surface Film Coefficient

1.3.2.3.1 Analytically

The few solvable cases learn that the convective surface film coefficient changes from spot to spot along a surface. For heat exchange, normal is to use a surface averaged value:

$$h_c = \frac{1}{A} \int h_{cA} \mathrm{d}A$$

If surface temperatures on specific spots are what matters, then the value there of the convective surface film coefficient should be used.

1.3.2.3.2 Numerically

Although CFD helps to calculate h_c as the results in Table 1.1 and Figure 1.24 show, it requires assuming a velocity distribution in the boundary layer.

Table 1.1 Convective surface film coefficient (h_e) against the envelope of a detached cube-shaped building: CFD-based correlations for a wind speed <15 m/s and 10 °C temperature difference with the air all around.

Emmel et al. (2007)	Surface to wind angle (°)	h_e, W/(m^2·K) (v_w = wind speed)
Outer walls	0	$5.15v_w^{0.31}$
	45	$3.34v_w^{0.34}$
	90	$4.78v_w^{0.71}$
	135	$4.05v_w^{0.77}$
	180	$3.54v_w^{0.76}$
Roofs	0	$5.11v_w^{0.78}$
	45	$4.6v_w^{0.79}$
	90	$3.76v_w^{0.85}$

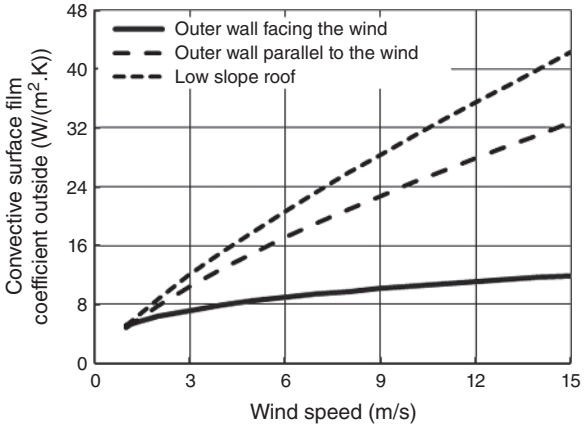


Figure 1.24 Cube-shaped building, CFD-based outside convective surface film coefficient.

1.3.2.3.3 Dimensionally

Many information on convection comes from experiments that help determining which dimensionless ratios between fluid properties, geometric data and kinematic characteristics require a same value in the experiments as in reality to allow extrapolation of the test results to reality. These ratios either follow from the differential equations or from Buckingham's π-theorem, stating that if a problem depends on n single-valued physical properties with p basic dimensions, then $n–p$ dimensionless ratios will fix the solution.

Describing forced convection so involves seven physical properties (L, λ_{fl}, v_{fl}, ρ_{fl}, μ_{fl}, c_{fl}, h_c (see above)) having four basic dimensions (representative length L, time t, mass M, temperature θ). So, three dimensionless ratios (π_1, π_2, π_3) are needed for a solution. They follow from turning $\pi = L^a\, \lambda_{fl}^b\, v_{fl}^c\, \rho_{fl}^d\, \mu_{fl}^e\, c_{fl}^f\, h_c^g$ into an equation containing the three ratios, written as $\pi_1 = f(\pi_2, \pi_3)$:

$$\pi = [L]^a \left[\frac{ML}{t^3\theta}\right]^b \left[\frac{L}{t}\right]^c \left[\frac{M}{L^3}\right]^d \left[\frac{M}{Lt}\right]^e \left[\frac{L^2}{t^2\theta}\right]^f \left[\frac{M}{t^3\theta}\right]^g$$

As the three must be dimensionless, the sum of the exponents per dimension should be 0, or:

for M			b			+	D	+	e			+	g	=	0
for L	a	+	b	+	c	−	$3d$	−	e	+	$2f$			=	0
for t		−	$3b$	−	c			−	e	−	$2f$	−	$3g$	=	0
for θ		−	b							−	f	−	g	=	0

The three so follow from recalculating the π-function three times, first for $g = 1$, $c = 0$, $d = 0$, then for $g = 0$, $a = 1$, $f = 0$ and finally for $g = 0$, $e = 1$, $c = 0$:

Solution 1 $a = 1, b = -1, e = 0, f = 0$ or $\pi_1 = \dfrac{h_c L}{\lambda_{fl}}$

Solution 2 $b = 0, c = 1, d = 1, e = -1$ or $\pi_2 = \dfrac{\rho_{fl} v_{fl} L}{\mu_{fl}}$

Solution 3 $a = 0, b = -1, d = 0, f = 1$ or $\pi_3 = \dfrac{c_{fl}\mu_{fl}}{\lambda_{fl}}$

Natural convection in turn demands four dimensionless ratios, of which two combine to one. The three for forced convection are called the Reynolds, Nusselt and Prandl number, while the three for natural convection are called the Grasshof, Nusselt and Prandl number:

$$\text{Reynolds}: \quad \text{Re} = v_{fl} L / \nu_{fl} \ (= \pi_2) \tag{1.62}$$

with v_{fl} the fluid's velocity, ν_{fl} the fluid's kinematic viscosity and L the characteristic length fixing the geometry (the hydraulic diameter for pipes, the dimension in the flow direction for walls and a calculated length for more complex cases). Reynolds so stands for the ratio between the inertia force and viscous friction. If low, friction gains and the result is laminar flow. If high, inertia wins and the result is turbulent flow. Or, the number fixes the flow type: laminar for $\text{Re} \leq 2000$, turbulent for $\text{Re} \geq 20\,000$, transient for $2000 < \text{Re} < 20\,000$.

$$\text{Nusselt}: \quad \text{Nu} = h_c L / \lambda_{fl} \ (= \pi_1) \tag{1.63}$$

with λ_{fl} the thermal conductivity of the fluid. The right-hand side can be rewritten as:

$$h_c L / \lambda_{fl} = (\text{grad}\,\theta)_s / [(\theta_{fl} - \theta_s)/L]$$

Or, Nusselt represents the ratio between the temperature gradient against a surface in the fluid and the mean temperature gradient along the characteristic length. A high value stands for the first large and the second small, the case with high fluid velocities. Physically, the number confirms that conduction governs the heat transfer against a surface. In fact, even for turbulent flow, a laminar boundary layer remains, however with a thickness that shrinks the higher the fluid velocity. The number underlines the importance of convection compared to conduction.

$$\text{Prandl}: \quad \text{Pr} = \nu_{fl} / a_{fl} \ (= \pi_3) \tag{1.64}$$

This number combines the heat and mass transfer by rating two analogous quantities: the thermal diffusivity, which determines how easy a local temperature change is spreading in the fluid, and the kinematic viscosity ν, which indicates how easy a local velocity change does it.

$$\text{Grasshof}: \quad \mathrm{Gr} = \beta_{\mathrm{fl}} g L^3 \Delta\theta / \nu_{\mathrm{fl}}^2 \tag{1.65}$$

with β_{fl} the volumetric expansion coefficient, g the acceleration by gravity and $\Delta\theta$ the representative temperature difference. The number replaces Reynolds in natural convection as the velocity then is the result of mainly temperature-induced density differences (βg), which simultaneously change the temperature field. This is why the terms L^3 and ν_{fl}^2 replace L and ν_{fl}, in the Reynolds number.

$$\text{Rayleigh:} \quad \mathrm{Ra} = \mathrm{GrPr} \tag{1.66}$$

This number replaces the product GrPr in the many natural convection formulae.

All experimental, numerical and analytical expressions for the convective surface film coefficient are now written as:

Natural convection	Mixed convection	Forced convection
$\mathrm{Nu} = c(\mathrm{Ra})_n$	$\mathrm{Nu} = F(\mathrm{Re}/\mathrm{Gr}^{1/2}, \mathrm{Pr})$	$\mathrm{Nu} = F(\mathrm{Re}, \mathrm{Pr})$

The coefficient c, the exponent n and $F(\ )$ differ among geometries, flow type and direction.

1.3.2.4 Values for the Convective Surface Film Coefficient

1.3.2.4.1 Flat Surfaces

For natural convection, L is the height for vertical surfaces, the side for square horizontal surfaces and the average of length and width for rectangular horizontal surfaces. The mean temperature between surface and undisturbed liquid or gas in turn fixes the value of the properties intervening. The relations advanced are:

	Conditions	Functions
Vertical surfaces	$\mathrm{Ra}_L \leq 10^9$	$\mathrm{Nu}_L = 0.56\ \mathrm{Ra}_L^{1/4}$
	$\mathrm{Ra}_L > 10^9$	$\mathrm{Nu}_L = 0.025\ \mathrm{Ra}_L^{2/5}$
Horizontal surfaces		
Heat flow upwards	$10^5 < \mathrm{Ra}_L \leq 2 \cdot 10^7$	$\mathrm{Nu}_L = 0.56\ \mathrm{Ra}_L^{1/4}$
	$2 \cdot 10^7 < \mathrm{Ra}_L < 3 \cdot 10^{10}$	$\mathrm{Nu}_L = 0.138\ \mathrm{Ra}_L^{1/3}$
Heat flow downwards	$3 \cdot 10^5 < \mathrm{Ra}_L < 10^{10}$	$\mathrm{Nu}_L = 0.27\ \mathrm{Ra}^{1/4}$

For forced convection, they are:

	Conditions	Functions
Laminar flow	$\mathrm{Pr} > 0.1$, $\mathrm{Re}_L < 5 \cdot 10^5$	$\mathrm{Nu}_L = 0.644\ \mathrm{Re}_L^{1/2} \mathrm{Pr}^{1/3}$
Turbulent flow	$\mathrm{Pr} > 0.5$, $\mathrm{Re}_L > 5 \cdot 10^5$	$\mathrm{Nu}_L = 0.036\ \mathrm{Pr}^{1/3} \left(\mathrm{Re}_L^{4/5} - 23\ \ 200 \right)$

Turning to buildings, air at atmospheric pressure and the ambient temperature are what counts. This simplifies the equations. For natural convection, the temperature difference $\Delta\theta$ between surface and undisturbed air is driving, as reflected by the relation heading the table:

$h_c = a(\Delta\theta/L)^b$	Conditions	a	b	L
Vertical surfaces	$10^{-4} < L^3 \Delta \mathrm{T} \leq 7$	1.4	1/4	Height
	$7 < \mathrm{L}^3 \Delta T \leq 10^3$	1.3	1/3	1
Horizontal surfaces				
Heat flow upwards	$10^{-4} < \mathrm{L}^3 \Delta \mathrm{T} \leq 0.14$	1.3	1/4	Eq. side
	$0.14 < \mathrm{L}^3 \Delta \mathrm{T} \leq 200$	1.5	1/3	1
Heat flow downwards	$2 \cdot 10^{-4} < \mathrm{L}^3 \Delta \mathrm{T} \leq 200$	0.6	1/4	1

In buildings with HVAC, the convective surface film coefficients for mixed convection are often related to the air change rate n, a number telling how many times an hour the air in a space is exchanged (ach). For rectangular spaces:

	Configuration	h_c (W/(m^2·K))
Walls	Forced convection, air diffusers at the ceiling, room isothermal	$-0.199 + 0.18n^{0.8}$
Floor		$0.159 + 0.116n^{0.8}$
Ceiling		$-0.166 + 0.484n^{0.8}$
Walls	Forced convection, air diffusers in the walls, room isothermal	$-110 + 0.132n^{0.8}$
Floor		$0.704 + 0.168n^{0.8}$
Ceiling		$0.064 + 0.00444n^{0.8}$

For forced convection outdoors, wind is the main actor, giving as relations:

Wind speed	Relation	Remarks
$v \leq 5$ m/s	$h_c = 5.6 + 3.9\,v$	For $v \leq 5$ m/s natural convection still intervenes, therefore the constant 5.6
$v > 5$ m/s	$h_c = 7.2\,v^{0.78}$	

That the value increases with wind speed, follows from a related drop in boundary layer thickness.

All relations given apply for air flowing along freestanding flat surfaces. Angles between two and corners between three surfaces disturb the flow. Moreover, if surfaces form a room, the overall flow pattern must satisfy the continuity equation. All this makes convection so complex that standards give constant average values:

EN-standard	Heat loss	Surface temperatures
Natural convection (= indoors)		
Vertical surfaces	3.5	2.5
Horizontal surfaces:		
Heat upwards (↑)	5.5	2.5
Heat downwards (↓)	1.2	1.2
Forced convection (= outdoors)	19.0	19.0

The reference temperature in indoor spaces is the air temperature, 1.7 m above the floor's centre. Outdoors, it's the air temperature measured by the nearest weather station. When calculating surface temperatures using local convective surface film coefficients, the reference should be the air temperature at the same spot just outside the boundary layer. Large temperature differences, complex geometries or inner surfaces hided by furniture require a more correct calculation. In such cases, the tabled formulae or data from literature should be used.

1.3.2.4.2 Cavities

The name cavity refers to an air layer between material layers with a width, small compared to its length or height, see Figure 1.25. The convective heat fluxes against the warm (1) and cold cavity face (2) equal:

$$q_{c1} = h_{c1}(\theta_{s1} - \theta_c) \qquad q_{c2} = h_{c2}(\theta_c - \theta_{s2})$$

In both, θ_c is the gas or air temperature in the centre of the cavity. If unvented, both fluxes must on average be equal, giving as mean value:

$$q_c = \frac{h_{c1} h_{c2}}{h_{c1} + h_{c2}} (\theta_{s1} - \theta_{s2}) \tag{1.67}$$

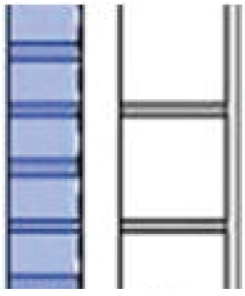


Figure 1.25 Wall with cavity between veneer and inside leaf.

Replacing both surface film coefficients by one value h_c, simplifies the formula to:

$$q_c = (h_c/2)(\theta_{s1} - \theta_{s2})$$

In reality, convection in a cavity includes conduction, reason why this relation is often rewritten as if the last dominates, be it with the λ_{fl}-value multiplied with the Nusselt number Nu:

$$q + q_c = (\lambda_{fl}\mathrm{Nu})\Delta\theta_s/d = h'_c\,\Delta\theta_s \tag{1.68}$$

Here, d is the cavity width (m) and $\Delta\theta_s$ the temperature difference between both cavity faces (°C). In infinitely long horizontal cavities, circular eddies, called Bénard cells, develop, while in infinitely high vertical cavities, air rotation does. In both cases, Nu to the width d writes as:

$$\mathrm{Nu}_d = \max\left[1, 1 + \frac{m\,\mathrm{Ra}_d^r}{\mathrm{Ra}_d + n}\right] \quad (10^2 \le \mathrm{Ra}_d \le 10^8) \tag{1.69}$$

In it, temperature difference between both cavity faces acts as reference for the Rayleigh number Ra_d, while cavity width (d) act as characteristic length and m, r and n have as values:

	m	***n***	***r***
Horizontal cavity			
Heat flow downwards	0		
Heat flow upwards	0.07	3200	1.33
Vertical and tilted cavity with slope above 45°			
	0.024	10 100	1.39
Tilted cavity with slope below 45°			
Heat flow upwards	0.043	4100	1.36
Heat flow downwards	0.025	13 000	1.36

Ra_d lower than 100 means pure conduction, $\mathrm{Nu}_d = 1$. Convection in finite cavities diverges strongly from convection in infinite ones. With d cavity width, H cavity height and L cavity length, all in m, with $^{\mathrm{lam}}$ the superscript for laminar, $^{\mathrm{turb}}$ the superscript for turbulent and $^{\mathrm{transient}}$ the superscript for transient, Nu_d becomes:

		Nu_d
		Vertical cavity
Rayleigh number upper limit value (Ra_{max}) for the applicability of Nu_d depending on the ratio H/d: H/d = 5 20 40 80 100 $Ra_{max} = 10^8\ 2.10^6\ 2.10^5\ 3.10^4\ 1.2\ 10^4$	$\max\left(Nu_d^{lam}, Nu_d^{turb}, Nu_d^{transient}\right)$, with $Nu_d^{lam} = 0.242\left(\frac{Ra_d d}{H}\right)^{0.273}$ $Nu_d^{turb} = 0.0.0605\ Ra_d^{0.33}$ $Nu_d^{transient} = \left[1 + \left(\frac{0.104 Ra_d^{0.293}}{1 + (6310/Ra_d)^{1.36}}\right)^3\right]^{0.33}$	
		Horizontal cavity
Heat flow upwards	$Ra_d \leq 1708$	1
	$Ra_d > 1708$	$\max\left[1.1537\ d^2\left(\frac{\Delta\theta}{L}\right)^{1/4}\right]$
Heat flow downwards		1
		Tilted cavity: see literature

1.3.2.4.3 Pipes

Experimental and semi-experimental work has been done on convection between a pipe's outer surface and the fluid or gas in the ambient around (Figure 1.26).

The result is a series of formulae. For natural convection:

Vertical pipe	$Ra_d \leq 10^9$	$Nu_L = 0.555\ Ra_d^{1/4}$
	$Ra_d > 10^9$	$Nu_L = 0.021\ Ra_d^{2/5}$
Horizontal pipe	$Ra_d \leq 10^9$	$Nu_L = 0.530\ Ra_d^{1/4}$

Figure 1.26 Convection around an insulated pipe.

For forced convection:

$Re_d < 500$	$Nu_d = 0.43 + 0.48\,Re_d^{1/2}$
$Re_d > 500$	$Nu_L = 0.46 + 0.00128\,Re_d$

In these equations, the characteristic length (d) is the pipe's outer diameter, while all properties are linked to the average temperature between the undisturbed fluid or gas around (θ_{fl}) and the pipe's outer surface (θ_s).

1.3.3 Radiation

1.3.3.1 In General

Radiation differs fundamentally from conduction and convection. In fact, what displaces heat is neither the kinetic energy of atoms and free electrons in a material nor the macroscopic movements in a gas or a liquid but the electromagnetic waves surfaces exchange. Each surface warmer than 0 K emits and absorbs electromagnetic waves, with those absorbed agitating the atoms and electrons in the material closest to that surface, that so is warming up.

Electromagnetic waves span a wide range of wavelengths (λ), with ultraviolet (UV), visible light (L) and infrared (IR) ($\lambda = 10^{-7}$ to $10^{—3}$ m) as the thermally active ones. Aside, the wavelength stands for the ratio between the speed of the photons propagated in m/s and the frequency in Hz, 1 Hz being 1 wavelength passing per second:

$\lambda^{(1)} \leq 10^{-6}$ µm	Cosmic radiation
$10^{-6} < \lambda \leq 10^{-4}$ µm	Gamma rays
$10^{-4} < \lambda \leq 10^{-2}$ µm	X-rays
$10^{-2} < \lambda \leq 0.38$ µm	UV
$0.38 < \lambda \leq 0.76$ µm	Visible light
$0.76 < \lambda \leq 10^{3}$ µm	IR
$\lambda > 10^{3}$ µm	Radio waves

Thermal radiation does not demand a medium. In fact, the transfer is most unhindered in vacuum, where it moves with a speed equal to 299 792.5 km/s. How much radiation material surfaces emit, depends on their nature and their temperature. A net heat exchange between surfaces anyhow requires them being at a different temperature.

1.3.3.2 Definitions

Table 1.2 outlines the way thermal radiation can be quantified. The related, so-called spectral values are given by the quantity considered per wavelength. A single wavelength emitted is giving monochromatic radiation, different wavelengths emitted together are giving coloured radiation.

Table 1.2 Radiant heat transfer, variables.

Variable	Definition (units)	Equations
Radiant heat Q_R	The heat emitted or received under the form of electromagnetic waves. Q_R is a scalar, units J	
Radiant heat flow Φ_R	The radiant heat per unit of time. Φ_R is a scalar, units W	$\frac{dQ_R}{dt}$
Radiant heat flux q_R	The radiant heat flow per unit of surface. As a surface emits radiation to and receives radiation from all directions, q_R, is a scalar, units W/m^2. The term irradiation, symbol E, is used for the incoming radiant heat flux, the term emittance, symbol M, for the one emitted	$\frac{d^2Q_R}{dA\,dt}$
Radiation intensity I	The radiant energy emitted in a specific direction. The intensity is a vector, units W/(m^2·rad), with dω the elementary angle in the direction considered	$\frac{dq_R}{d\omega}$ or $\frac{d^2\Phi_R}{dA\,d\omega}$
Luminosity L	The ratio between the radiant heat flow rate in a direction ϕ and the apparent surface, seen from that direction. The luminosity is a vector, units W/(m^2·rad). It describes how a receiving surface sees one emitting.	$\frac{d^2\Phi_R}{\cos(\phi)\,dA\,d\omega}$

1.3.3.3 Reflection, Absorption and Transmission

When a radiant flux (q_{Ri}) emitted by a surface at temperature T touches another, a part is absorbed (q_{Ra}), a part is reflected (q_{Rr}) and, if transparent, a part is transmitted (q_{Rt}):

$$\alpha = q_{Ra}/q_{Ri} \quad \rho = q_{Rr}/q_{Ri} \quad \tau = q_{Rt}/q_{Ri} \tag{1.70}$$

In there, α, ρ and τ are the average absorptivity, reflectivity and transmissivity at the temperature of the surface touched. Conservation of energy now states that the sum of these three must be 1, or:

$$\alpha + \rho + \tau = 1$$

Sum 1 however does not hold for radiation at different temperatures.

A distinction must also be made between diffuse and specular reflection. The last obeys the laws governing optics: incident and reflected beam in the same plane perpendicular to the surface and the angles of incidence ϕ_i and reflection ϕ_r equal (Figure 1.27). Most surfaces however reflect incident radiation diffuse, in all directions.

The reflectivity in direction α can be defined opposite the radiant intensity touching the surface under an angle ϕ ($I_{Ri\phi}$):

$$\rho_\varphi = I_{Rr\alpha}/I_{RiRi\varphi}$$

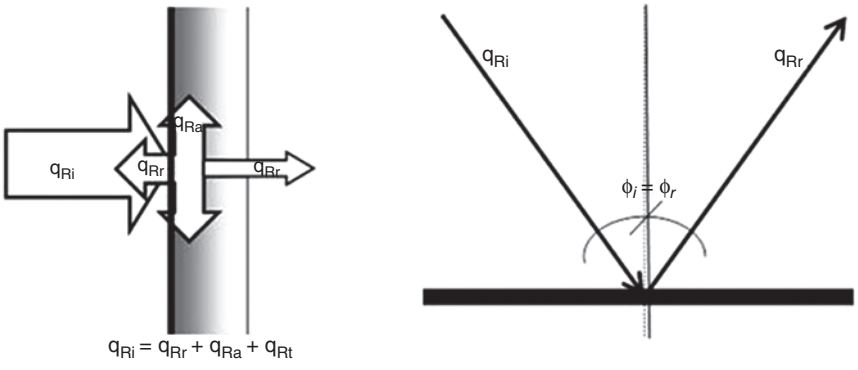


Figure 1.27 Reflection, absorption and transmission at a surface, specular reflection.

In it, $I_{\mathrm{Rr}\alpha}$ is the reflected intensity in direction α. For specular reflecting surfaces, the reflectivity is:

$$\rho_\varphi = (I_{\mathrm{Rr}}/I_{\mathrm{Ri}})_\varphi$$

with ρ_ϕ function of the angle of incidence.

Most building and insulating materials are opaque for thermal radiation ($\tau = 0$). What comes in is absorbed in a thin surface layer, 10^{-6} m thick for metals and 10^{-4} m thick for other materials. Because of that, the term absorbing surface is commonly used. Instead, most gases, liquids and some solids such as glass and many synthetics are selectively transparent but also show selective absorption in their mass depending on their extinction coefficient (a), (Figure 1.28):

$$\frac{\mathrm{d}q_R}{q_R} = -a\,\mathrm{d}x \tag{1.71}$$

For such material layer with thickness d, the transmitted and absorbed radiant fluxes so become:

$$q_{\mathrm{Rt}} = q_{\mathrm{Ri}} \exp(-ad) \quad q_{\mathrm{Ra}} = q_{\mathrm{Ri}} - q_{\mathrm{Rt}} = q_{\mathrm{Ri}}[1 - \exp(-ad)]$$

with q_{Ri} the incoming radiant flux. Absorptivity and transmissivity consequently equal:

$$\alpha = 1 - \exp(-ad) \quad \tau = \exp(-ad)$$

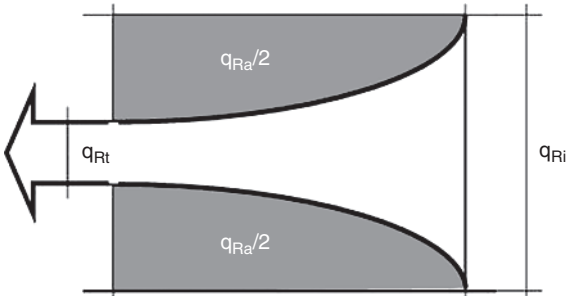


Figure 1.28 Transparent materials, the way they absorb radiation passing through.

For an irradiated surface of a transparent material, the specular reflectivity so looks:

$$\rho = \frac{I_r}{I_i} = \left[\frac{n_1 \cos(\phi_i) - n_2 \cos(\phi_t)}{n_1 \cos(\phi_i) + n_2 \cos(\phi_t)}\right]^2$$

with n_1 the refractive index of the gas crossed, n_2 the refractive index of the material, ϕ_i the angle of incidence in the gas and ϕ_t the angle of transmittance in the transparent material.

As mentioned, absorptivity, reflectivity, and transmissivity vary with temperature, thus, with wavelength, albeit for the radiation intensity also the angle of incidence matters. This dependency anyhow can change quickly. Take glass. Its transmissivity for visible light is large whereas for low temperature IR it nears zero with the absorptivity then passing 0.9. This explains the greenhouse effect. All surfaces in a space with glazing absorb the short-wave, high temperature solar radiation transmitted by that glazing and re-emits it as low temperature IR radiation, which the glass absorbs. This turns conduction through the glass into the only way to temper warming indoors. Also, part of the absorbed IR radiation by the glass is released to the indoors, so, both may turn the indoors uncomfortably warm. Transparent synthetics behave the same way, be it that some also transmit IR.

1.3.3.4 Radiant Surfaces

1.3.3.4.1 Types

A distinction is made between black surfaces absorbing all incident radiation ($\alpha = 1$, $\rho = 0$, $\tau = 0$, $\alpha \neq f(\lambda, \phi)$), grey surfaces showing a constant absorptivity <1, white surfaces with absorptivity 0 and coloured surfaces, whose absorptivity depends on temperature and the direction of incidence. Although fictitious, real surfaces are considered grey. In order to falsify reality not too much, short wave solar radiation, subscript S, and long wave radiation, subscript L, with their absorptivity and reflectivity, are kept separate.

1.3.3.4.2 Black Surfaces

Independent of temperature, black surfaces have an emissivity 1. In fact, according to the second law of thermodynamics, if at different temperature in a closed system, they must evolve towards temperature equilibrium. Once there, all emit and absorb the same amount of radiation, meaning absorptivity equals emissivity, thus 1. What concerns the direction radiation is emitted, black bodies obey Lambert's law: luminosity constant. Or, the radiant intensity is (Figure 1.29):

$$I_{b\phi} = L_b \cos(\phi) \tag{1.72}$$

This equation is known as the cosine law. It offers a simple relation between emittance M_b and luminosity. From the definitions in Table 1.2, a full hemisphere integration gives:

$$M_b = L_b \int_\omega \cos(\varphi)\ \mathrm{d}\omega$$

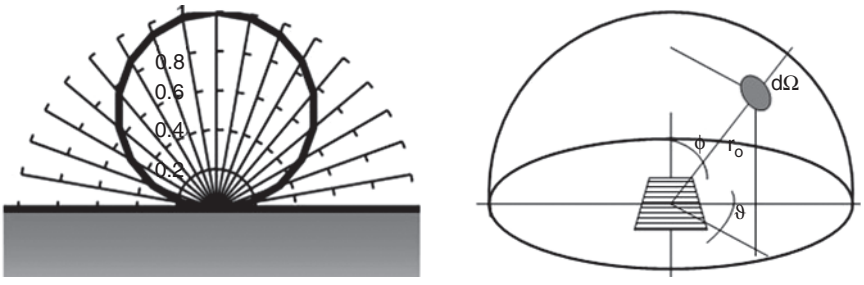


Figure 1.29 (a) Thick line showing the effect of the cosine law on radiation intensity, (b) proving the law.

The angle dω is calculated assuming the hemisphere with radius r_0, surrounds the infinitesimal surface dA with luminosity L_b completely (Figure 1.29). Or, dω equals:

$$\mathrm{d}\omega = r_0^2 \sin(\varphi)\mathrm{d}\varphi\,\mathrm{d}\vartheta$$

On that hemisphere, the intensity drops to $I_{b\varphi}/r_0^2$, or:

$$M_\mathrm{b} = L_\mathrm{b}\int_0^{2\pi}\int_0^{\pi/2}\frac{\cos(\varphi)}{r_0^2}r_0^2\sin(\varphi)\mathrm{d}\varphi\,\mathrm{d}\vartheta = [-\pi L_\mathrm{b}\cos^2(\varphi)]_0^{\pi/2} = \pi L_\mathrm{b} \qquad (1.73)$$

According to Planck's law, the spectral density of the emittance is:

$$M_{\mathrm{b}\lambda} = \frac{2\pi c^2 h\lambda^{-5}}{\exp\left(\dfrac{ch}{k\lambda T}\right) - 1} \qquad (1.74)$$

with c the speed of the electromagnetic waves in m/s, h Planck's constant ($6.624 \cdot 10^{-34}$ J·s) and k the Boltzmann constant ($1.38047 \cdot 10^{-23}$ J/K). The products $2\pi c^2 h$ and ch/k figure as radiation constants for black bodies, symbols C_1(=$3.7415 \cdot 10^{-16}$ W·m^2) and C_2 (=$1.4388 \cdot 10^{-2}$ m·K). Figure 1.30 shows the spectral density of the emittance for a few absolute temperatures.

The emittance, given by the surface bounded by the curves, increases quickly with temperature, while the maximum occurs at ever shorter wavelengths. In the

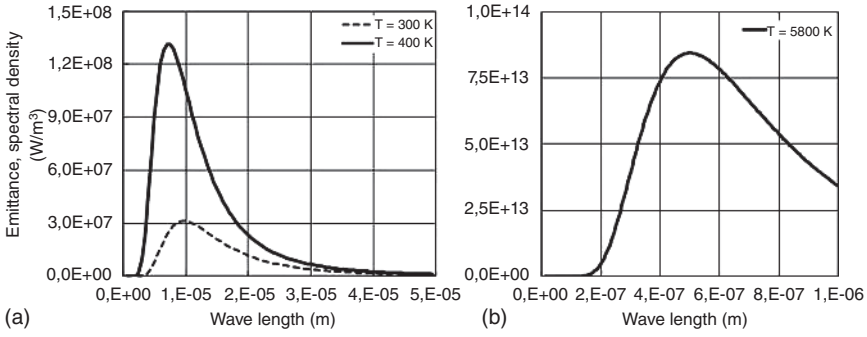


Figure 1.30 Black surface emittance, (a) spectral density at ambient and higher temperature, (b) the same for the sun as black surface.

$[\lambda, M_{B\lambda}]$-plane their geometric locus is a hyperbole of the fifth order, while the wavelengths obey Wien's law:

$$\lambda_M T = 2898 \ (\lambda_M \text{ in } \mu\text{m}) \tag{1.75}$$

At 20 °C, $\lambda_M = 2898/293.15 = 9.9\ \mu$m, a maximum in the IR part. For the sun, with a radiant temperature of 5800 K, $\lambda_M = 2898/5800 = 0.5\ \mu$m, a maximum in the midst of the visible light.

The emittance M_b now follows from integrating Planck's law over the wavelength:

$$M_b = \int_0^\infty M_{b\lambda}\, d\lambda = \frac{2\pi^5 k^4}{15c^2h^3} T^4 = \sigma T^4 \tag{1.76}$$

a result, known as the Stefan–Boltzmann law with σ Stefan's constant, $5.67 \cdot 10^{-8}\ \text{W/(m}^2\cdot\text{K}^4)$. With Wien, Stefan–Boltzmann preceded Planck's law, which needed quantum mechanics to get formulated. Stefan–Boltzmann is mostly written as:

$$M_b = C_b \left(\frac{T}{100}\right)^4 \tag{1.77}$$

with C_b the black surface constant, $5.67\ \text{W/(m}^2\cdot\text{K}^4)$, and $T/100$ the reduced radiant temperature in K. The luminosity and radiation intensity so become:

$$L_b = \frac{M_b}{\pi} = \frac{C_b}{\pi}\left(\frac{T}{100}\right)^4 \quad I_{b\phi} = \frac{d^2\Phi_{Rb}}{dA\, d\omega} = L_b \cos(\phi) = \frac{C_b}{\pi}\left(\frac{T}{100}\right)^4 \cos(\phi)$$

When two infinitely small black surfaces dA_1 and dA_2, separated by a transparent gas, see each other as in Figure 1.31, the elementary radiant heat flow from dA_1 to dA_2 becomes:

$$d^2\Phi_{R,1\to 2} = I_{b1}\, dA_1\, d\omega_1 = \frac{M_{b1}}{\pi} \cos(\phi_1) dA_1\, d\omega_1$$

with $d\omega_1$ the angle under which dA_1 sees dA_2:

$$d\omega_1 = dA_2 \cos(\varphi_2)/r^2$$

This allows rewriting the flow exchanged:

$$d^2\Phi_{R,1\to 2} = \frac{M_{b1}}{\pi} \cos(\phi_1)\cos(\phi_2) dA_1 \frac{dA_2}{r^2}$$

Vice versa, the elementary heat flow from dA_2 to dA_1 is:

$$d^2\Phi_{R,2\to 1} = \frac{M_{b2}}{\pi} \cos(\phi_1)\ \cos(\phi_2)\, dA_1 \frac{dA_2}{r^2}$$

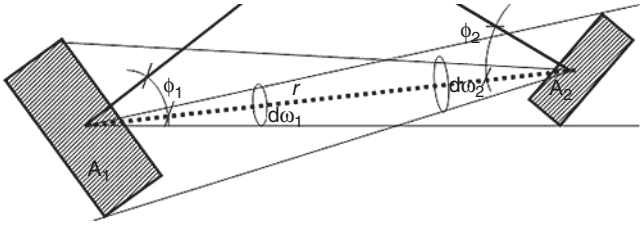


Figure 1.31 View factor between the surfaces A_1 and A_2.

The resulting radiant heat flow between the two so becomes:

$$\text{From } dA_1 \text{ to } dA_2: \quad d^2\Phi_{R,12} = d^2\Phi_{R,1\to 2} - d^2\Phi_{R,2\to 1}$$
$$= \frac{(M_{b1} - M_{b2})\cos(\phi_1)\cos(\phi_2)dA_1\, dA_2}{\pi r^2}$$
$$\text{From } dA_2 \text{ to } dA_1: \quad d^2\Phi_{R,21} = d^2\Phi_{R,2\to 1} - d^2\Phi_{R,1\to 2}$$
$$= \frac{(M_{b2} - M_{b1})\cos(\varphi_1)\cos(\varphi_2)dA_1\ \ dA_2}{\pi r^2}$$

If both surfaces have finite dimensions, the exchange looks:

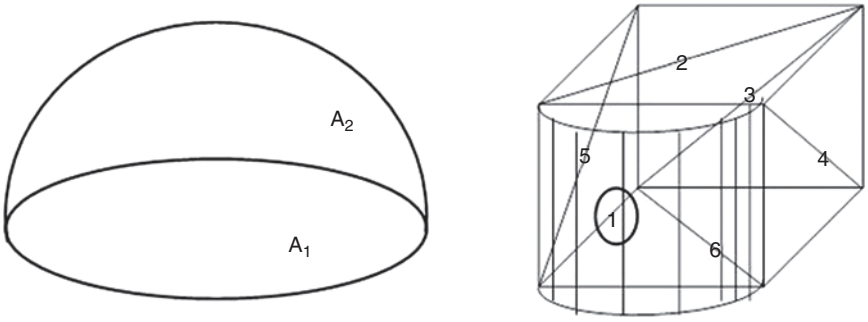
$$\text{From } A_1 \text{ to } A_2: \ \Phi_{R,12} = (M_{b1} - M_{b2})A_1 \left[\frac{1}{\pi A_1}\int_{A_1}\int_{A_2} \frac{\cos(\varphi_1)\cos(\varphi_2)dA_2\, dA_1}{r^2}\right] \tag{1.78}$$

$$\text{From } A_2 \text{ to } A_1: \ \Phi_{R,21} = (M_{b2} - M_{b1})A_2 \left[\frac{1}{\pi A_2}\int_{A_2}\int_{A_1} \frac{\cos(\varphi_1)\cos(\varphi_2)dA_1\, dA_2}{r^2}\right] \tag{1.79}$$

The terms between brackets in both are called the view factor, symbol F. Other names are angle factor, shape factor or configuration factor. If surface A_1 is the emitting, the view factor writes as F_{12}, if it's A_2, the view factor writes as F_{21}. View factors indicate which fraction of the radiant flow emitted by one surface reaches a seen other. Their size, form, distance, and viewing angle define the value, 1 if all emitted radiation reaches the other.

Looking to the properties of view factors, first comes their reciprocity: $A_1F_1 = A_2F_2$. This follows from the definition. If a surface A_2 surrounds A_1, the view factor from A_1 to A_2 turns 1 (Figure 1.32), a result that holds for any surface surrounded by $n-1$ other surfaces, so forming a closed volume:

$$\sum_{j=2}^{n} F_{1j} = 1$$



$F_{12} = 1$ (a)

$F_{11} + F_{12} + F_{13} + F_{14} + F_{15} + F_{16} = 1$ (b)

Figure 1.32 View factor between surface 1 (a) completely surrounded by surface 2 or (b) by the surfaces 2–6. In (b) surface 1 also radiates to itself.

A view factor 1 also holds for two infinitely mutually parallel surfaces, take both end faces of a cavity. For a few other simple surface configurations, the view factor can be calculated analytically. A point situated at a distance D from a rectangle with sides L_1 and L_2, surface $L_1 \cdot L_2 = A_2$, has a view factor with that rectangle equal to the ratio between the angle, under which it sees the rectangle, and the 4π of the sphere surrounding the point:

$$F_{12} = \frac{1}{4\pi}\int_{A_2} \frac{\cos(\phi)}{r^2} \mathrm{d}A_2$$

For a point above a corner of that rectangle, the formula turns into ($\cos(\varphi) = D/r$, $r^2 = D^2 + x^2 + y^2$):

$$F_{12} = \frac{1}{4\pi}\int_0^{L_1}\int_0^{L_2} \frac{D}{(D^2 + x^2 + y^2)^{3/2}} \mathrm{d}y\,\mathrm{d}x \text{ giving } F_{12} = \frac{1}{8} - \frac{1}{4\pi} a\tan\left(\frac{D\sqrt{L_1^2 + L_2^2}}{L_1 L_2}\right)$$

Other positions of the point can be converted to the corner case by using the approach shown in Figure 1.33, giving as view factor:

$$F_{12} = F_{1a} + F_{1b} + F_{1c} + F_{1d}$$

Radiation between the human head and a ceiling is an example of such point to surface situation. Another case concerns an infinitesimal small surface $\mathrm{d}A_1$ parallel to but at a distance D of a rectangle with sides L_1 and L_2 and surface $L_1 \cdot L_2 = A_2$. The formula for the view factor then is:

$$F_{12} = \frac{1}{\mathrm{d}A_1}\int_{\mathrm{d}A_1}\int_{A_2} \frac{\cos(\phi_1)\cos(\phi_2)\mathrm{d}A_2\,\mathrm{d}A_1}{\pi r^2}$$

The way $\mathrm{d}A_1$ sees each infinitesimal surface $\mathrm{d}A_2$ in A_2 is independent of $\mathrm{d}A_1$'s position, or:

$$F_{12} = \frac{1}{\pi}\int_{A_2 \text{ seen by } \mathrm{d}A_1} \frac{\cos(\phi_1)\cos(\phi_2)\mathrm{d}A_2}{r^2}$$

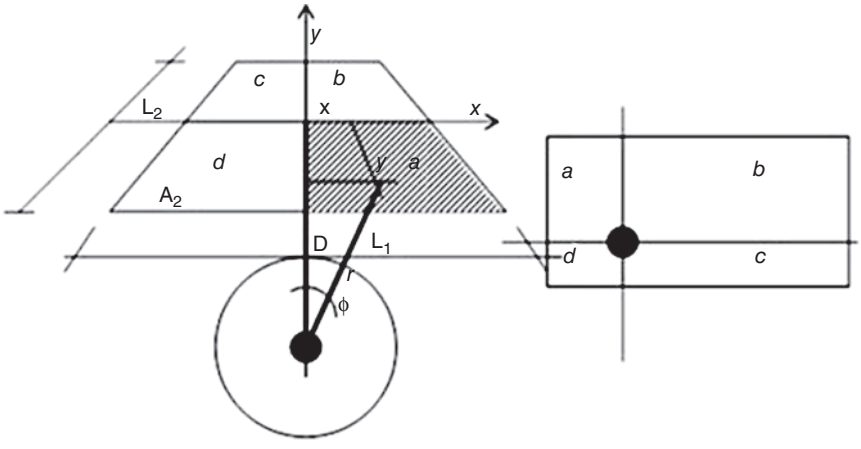


Figure 1.33 Left view factor between a point at distance D of surface A_2.

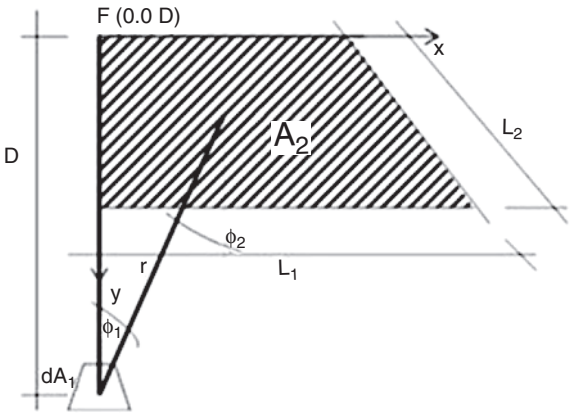


Figure 1.34 View factor between an infinitesimal small surface dA_1 parallel to and at a distance D from the corner (0,0) of surface A_2.

With $\mathrm{d}A_1$ situated above the corner (0, 0) of the rectangle A_2 (Figure 1.34), the view factor becomes:

$$F_{12} = \frac{1}{\pi}\int_0^{L_1}\int_0^{L_2} \frac{\cos(\phi_1)\cos(\phi_2)}{r^2}\mathrm{d}x\,\mathrm{d}y$$

With $\cos\phi_1 = \cos\phi_2 = D/r$ and $r^2 = D^2 + x^2 + y^2$, this equation simplifies to:

$$F_{12} = \frac{1}{\pi}\int_0^{L_1}\int_0^{L_2} \frac{D^2\mathrm{d}y\,\mathrm{d}x}{(D^2 + x^2 + y^2)^2}$$

with as solution:

$$F_{12} = \frac{1}{2\pi}\left[\frac{L_1}{\sqrt{D^2 + L_1^2}}a\tan\left(\frac{L_2}{\sqrt{D^2 + L_1^2}}\right) + \frac{L_2}{\sqrt{D^2 + L_2^2}}a\tan\left(\frac{L_1}{\sqrt{D^2 + L_2^2}}\right)\right]$$

Other configurations can be solved numerically, an example being an infinitely small surface $\mathrm{d}A_1$ standing at a distance D perpendicular to a rectangle with sides L_1 and L_2 and surface $L_1 \cdot L_2 = A_2$. Then, $\mathrm{d}A_1$ does not see the part of A_2, which extends behind the intersection with the plane to which it belongs. The numerical formula used is:

$$F_{12} = \frac{D\Delta x\Delta y}{2\pi}\left[\sum_{x=\Delta x/2 \text{ to } L_1-\Delta x/2 \text{ step } \Delta x}\ \sum_{y=\Delta y/2 \text{ to } L_2-\Delta y/2 \text{ step } \Delta y} \frac{\sqrt{x^2 + y^2}}{(x^2 + y^2 + D^2)^2}\right]$$

Common in beam-shaped rooms are three pairs of equally large parallel surfaces, two for the walls and one for the floor and ceiling. The three are perpendicular to each other and have common edges and corners. An analytical calculation of their view factors is not doable but a numerical solution is easily programmed on a spreadsheet.

Included the view factor, the radiant heat flows and fluxes write as:

$$\begin{aligned} \phi_{\mathrm{R},12} &= A_1F_{12}(M_{\mathrm{b}1} - M_{\mathrm{b}2}) & q_{\mathrm{R},12} &= F_{12}(M_{\mathrm{b}1} - M_{\mathrm{b}2}) \\ \phi_{\mathrm{R},21} &= A_2F_{21}(M_{\mathrm{b}2} - M_{\mathrm{b}1}) & q_{\mathrm{R},21} &= F_{21}(M_{\mathrm{b}2} - M_{\mathrm{b}1}) \end{aligned} \qquad (1.80)$$

For a number of black surfaces radiating to each other, the flow and flux per surface so equal:

$$\phi_{R1n} = A_1 \sum_{j=2}^{n} [F_{1j}(M_{b1} - M_{bj})] q_{R1n} = \sum_{j=2}^{n} [F_{1j}(M_{b1} - M_{bj})] \tag{1.81}$$

with $j = 1$ the surface considered and $j = 2$ to $=n$ the $n - 1$ other.

1.3.3.4.3 Grey Surfaces

The laws governing radiation between grey surfaces are similar to the black surface ones, except for the radiant exchange. Per wavelength and direction, grey surfaces emit a constant fraction of the black. Related ratio is called the emissivity (e). As conservation of energy imposes that absorptivity (α) and emissivity must be equal, the reflectivity (ρ) becomes:

$$\rho = 1 - \alpha = 1 - e$$

Grey surfaces with reflectivity 1 are called white. As the radiant heat flux must follow the cosine law, their emittance equals:

$$M = \pi L$$

The spectral density follows from Planck's law multiplied with the emissivity e, giving as total emittance:

$$M = e\, C_b \left(\frac{T}{100} \right)^4 \tag{1.82}$$

Grey surfaces reflect part of the incident radiation. Is eM_b the one's emittance and E the irradiation by the other, then the radiosity of that one is:

$$M' = eM_b + \rho E \tag{1.83}$$

The difference between radiosity and irradiation now fixes the flux emitted:

$$q_R = M' - E \tag{1.84}$$

Eliminating the irradiation E from both equations so gives:

$$q_R = M' - \frac{M' - e\, M_b}{\rho} = -\frac{e}{\rho}(M' - M_b) \tag{1.85}$$

The radiant flux received thus is:

$$q_R = \frac{e}{\rho}(M' - M_b) \tag{1.86}$$

Otherwise said, a grey surface behaves as a black, be it with a grey filter in front and a radiant resistance given by the ratio of the grey reflectivity to its emissivity (ρ/e) in between that filter with a radiosity M' and a fictitious black surface with emittance M_b. For two grey surfaces A_1 and A_2, separated by a transparent gas-like air, the radiant flow exchanged so is:

From fictitious black surface 1 to grey filter 1: $\Phi_{R,11} = \frac{e_1}{\rho_1} \left(M_{b1} - M'_1 \right) A_1$

From grey filter 1 to grey filter 2: $\Phi_{R,12} = F_{12} \left(M'_1 - M'_2 \right) A_1$

From grey filter 2 to fictitious black surface 2: $\Phi_{R,22} = \frac{e_2}{\rho_2} \left(M'_2 - M_{b2} \right) A_2$

From grey filter 1 to grey filter 2 the radiant heat flow looks like the one between two black surfaces. Indeed, the emittance of a black and the radiosity of a grey both obey Lambert's law for diffuse radiation. As the three flows now must be equal, elimination of the unknown radiosities M_1' and M_2' gives as flow from grey surface A_1 to grey surface A_2 and vice versa:

$$\Phi_{R,12} = \left[\frac{1}{\frac{\rho_1}{e_1} + \frac{1}{F_{12}} + \frac{\rho_2 A_1}{e_2 A_2}} \right] (M_{b1} - M_{b2})\, A_1$$

$$\Phi_{R,21} = \left[\frac{1}{\frac{\rho_2}{e_2} + \frac{1}{F_{21}} + \frac{\rho_1 A_2}{e_1 A_1}} \right] (M_{b2} - M_{b1})\, A_2 \tag{1.87}$$

The term between brackets is called the radiation factor F_R, written as $F_{R,12}$ if A_1 is the emitter and $F_{R,21}$ if it's A_2. Dividing both equations by the surface of the emitter gives the fluxes.

A simple configuration is an infinite cavity having two isothermal end faces at different temperature. Then, $F_{12} = F_{21} = 1, A_1 = A_2$, and the flux exchanged becomes ($\rho = 1 - e$):

$$q_{R,12} = \frac{\Phi_{R,12}}{A_1} = \left[\frac{1}{\frac{1}{e_1} + \frac{1}{e_2} - 1} \right] (M_{b1} - M_{b2}) \tag{1.88}$$

The term between brackets represents the radiation factor between both faces. Is one white, face 1, then $F_{R,12} = 1/(1/0 + 1/e_2 - 1) = 0$. Is one black, face 2, then $F_{R,12} = 1/(1/e_1 + 1/1 - 1) = e_1$

Another configuration is an isothermal surface A_1 surrounded by another isothermal surface A_2 at different temperature. F_{12} then is 1 and the radiant flow becomes:

$$\Phi_{R,12} = \frac{e_1 e_2}{e_2 \rho_1 + e_1 e_2 + \frac{e_1 \rho_2 A_1}{A_2}} (M_{b1} - M_{b2})\, A_1$$

Are both nearly black ($e > 0.9$), then the denominator nears 1 and:

$$\Phi_{R,12} = e_1 e_2 (M_{b1} - M_{b2}) A_1.$$

More, if A_1 is very small compared to A_2, giving a ratio ≈ 0, the equation further simplifies to:

$$\Phi_{R,12} = e_1 (M_{b1} - M_{b2}) A_1 \tag{1.89}$$

Or, the radiant heat flow exchanged then only depends on the emissivity of the surrounded surface A_1. When multiple isothermal grey surfaces, all at different but uniform temperature, see each other, the radiant flow between one (A_1) and the $n-1$ other writes as:

From black 1 to grey filter 1: $\Phi_{R,11} = \frac{e_1}{\rho_1}\left(M_{b1} - M'_1\right) A_1$

From grey filter 1 to the $n-1$ other grey filters: $\Phi_{R,1\,\text{to}\,j} = \sum_{j=2}^{n} F_{1j}\left(M'_1 - M'_j\right) A_1$

As both are equal, the black emissivity becomes: $M_{b1} = \left(1 + \frac{\rho_1}{e_1}\sum_{j=2}^{n} F_{1j}\right) M'_1 - \frac{\rho_1}{e_1}\sum_{j=2}^{n}\left(F_{1j} M'_j\right)$

For the 2 to n surfaces surrounding, this equation converts to:

$$M_{b1} = \frac{M'_1}{e_1} - \frac{\rho_1}{e_1}\sum_{j=2}^{n}\left(F_{1j} M'_j\right) \tag{1.90}$$

With the radiosities M'_j of the grey surfaces unknown and the black emittances M_{b1} known, the result is a system of n equations with: n unknowns:

$$[M_{bj}]_n = [F]_{n.n}\left[M'_j\right]_n \tag{1.91}$$

In it, $[F]_{n.n}$ is the radiation matrix of the n isothermal grey surfaces. Solving the system gives the radiosities M'_j as function of the black surface emittances M_{bj}. The radiant flows then follow from inserting the M'_j's in the equations given for the heat exchange between these fictitious black surfaces and the grey filter in front.

1.3.3.4.4 Coloured Surfaces

For coloured surfaces, emissivity, absorptivity and reflectivity change with wavelength, so with temperature, sometimes even with the direction of the incident or emitted radiation. Kirchhoff's law ($e = \alpha$) still applies but Lambert's law does not as it requires a direction-independent emission. The spectral emittance per wavelength differs from black, though the ratio between coloured and black at a same temperature still fixes the emissivity of the coloured surface at that temperature.

To simplify things, coloured surfaces are considered grey, be it with a temperature-dependent emissivity. For the emittance and irradiance at strongly differing temperatures, take the terrestrial versus the solar temperature, Kirchoff's law no longer applies since the solar short-wave absorptivity (α_S) can be very different from the terrestrial long wave emissivity (e_L), an example being polished aluminium with $\alpha_S = 0.2$–0.4 versus $e_L = 0.05$.

1.3.3.5 Simple Formulae

At first sight, thermal radiation looks accessible. However, calculating all angle factors is cumbersome, while the system of equations, when multiple grey surfaces interfere, can be very large. A simpler approach so is welcomed. First, reality is reduced to two radiant surfaces: the considered surface 1 and the $n-1$ other, seen as one, the environment. Then, the environment is assumed black with as uniform radiant temperature θ_r the one of a black surface exchanging the same radiant flow with surface 1 as in reality. Solving the system of equations for all surfaces present then gives the radiosity of surface 1 (M'_1) as linear combination of the black body emittances of all n surfaces present:

$$M_1' = \sum_{i=1}^{n} a_{ri} M_{bi}$$

Insertion in the equation for the grey surface radiant flow received and equating with the radiant flow found for the surrounded surface 1, which is usually small compared to the environment, fixes that radiant temperature θ_r:

$$\theta_r = \sqrt[4]{\frac{1}{\rho_1}\left(\sum_{i=1}^{n} a_{ri} T_i^4 - e_1 T_1^4\right)} - 273.15$$

In analogy with convection, the radiant flux and flow to surface 1 now are written as:

$$q_r = h_r(\theta_{s1} - \theta_r) \qquad \Phi_r = h_r(\theta_{s1} - \theta_r)A_1 \tag{1.92}$$

In both, h_r is the surface film coefficient for radiation (W/(m^2·K)), which varies with the configuration considered, and θ_{s1} is the temperature of the receiving surface 1. If the environment surrounds surface 1, then the surface film coefficient for radiation follows from equating this simple flux equation to the one derived for surface 1 being small compared to the surrounding environment:

$$h_r = eC_b \left[\frac{\left(\frac{T_{s1}}{100}\right)^4 - \left(\frac{T_r}{100}\right)^4}{\theta_{s1} - \theta_r} \right] \tag{1.93}$$

The term between brackets is called the temperature ratio for radiation F_T:

$$F_T = \frac{T_m}{5000}\left[\left(\frac{T_{s1}}{100}\right)^2 + \left(\frac{T_r}{100}\right)^2\right] \approx \frac{4}{100}\left(\frac{T_m}{100}\right)^3 \tag{1.94}$$

As this ratio hardly varies for temperatures between −10 and 50 °C, the expression at the right side, making the flux quasi linear, suffices. This gives for the surface film coefficient:

$$h_r = e_1 C_b F_T \tag{1.95}$$

For two isothermal parallel surfaces, one at temperature θ_{s1}, and the other at temperature θ_{s2}, no detour via the radiant temperature is needed. The surface film coefficient is directly written as:

$$q_r = h_r(\theta_{s1} - \theta_{s2}) \quad \text{with} \quad h_r = \frac{5.67\, F_T}{\frac{1}{e_1} + \frac{1}{e_2} - 1}$$

If part of the surrounding surfaces has the same temperature as surface 1, the radiant temperature (θ_r') then only includes those at different temperature, giving as surface film coefficient:

$$h_r = \frac{e_1\, C_b\, F_{12}\, F_T}{e_1 + \rho_1 F_{12}}$$

with F_{12} the view factor between surface 1 and that part of the surroundings at different temperature. If surface 1 is almost black, the denominator turns 1 and:

$$h_r = e_1 \, C_b \, F_{12} \, F_T = 5.67 e_1 \, F_{12} \, F_T \tag{1.96}$$

In case two identical outer walls, equally warm, form a corner, the radiant exchange is restricted to half the environment with surfaces at different temperature. The view factor then is 0.5 and the surface film coefficient for radiation becomes:

$$h_r = e_1 \, C_b \, F_T / 2 = 2.84 \;\; e_1 \, F_T \tag{1.97}$$

Of course, the one equally warm can be included in the radiant temperature. The view factor then remains 1 but the radiant temperature changes.

1.4 Building-related Applications

1.4.1 Surface Film Coefficients and Reference Temperatures

1.4.1.1 Methodology

In most of the applications, conduction, convection and radiation mix up. Through any assembly without cavity, heat conduction is the mover, while between indoors and the inner face and outdoors and the outer face convection and radiation take over. Related heat fluxes write as the difference between the surface temperature and a reference temperature for the mode and ambient considered, multiplied by a surface film coefficient (h_c, h_r). Yet, because convection and radiation are coupled, a common ambient temperature replacing the mode linked ones is used, while the surface film coefficients in- and outdoors are both combined to one, h_i and h_e. Of course, if needed, the two modes can remain split, which for complex cases can be necessary. Then, as well an air as a radiant temperature intervenes in- and outdoors, while both modes keep their own surface film coefficient.

1.4.1.2 Indoors

For a surface at temperature θ_{si}, the convective heat flux with the air is:

$$[q_{ci}] = h_{ci,s}(\theta_{i,ob} - \theta_{si})$$

In it, $h_{ci,s}$ is the area averaged convective surface film coefficient and $\theta_{I,ob}$ the average air temperature backing the boundary layer. If not this, but the air temperature in the space's centre, 1.7 m above floor level (θ_i), is taken as reference, that flux equation changes to:

$$q_{ci} = h_{ci}(\theta_i - \theta_{si})$$

with h_{ci} the average convective surface film coefficient linked to the new reference temperature:

$$h_{ci} = h_{ci,s} \left(\frac{\theta_{i,ob} - \theta_{si}}{\theta_i - \theta_{si}} \right) \tag{1.98}$$

The radiant heat flux touching the surface in turn equals:

$$q_{ri} = h_{ri}(\theta_{ri} - \theta_{si})$$

with θ_{ri} the radiant temperature indoors. The total heat flux at the surface so becomes:

$$q_i = q_{ci} + q_{ri} = h_{ci}(\theta_i - \theta_{si}) + h_{ri}(\theta_{ri} - \theta_{si}) = (h_{ci} + h_{ri})\left[\underbrace{\left(\frac{h_{ci}\theta_i + h_{ri}\theta_{ri}}{h_{ci} + h_{ri}}\right)}_{\theta_{ref,i}} - \theta_{si}\right] \tag{1.99}$$

The sum $h_{ci} + h_{ri}$ is called the inside surface film coefficient, symbol h_i, units W/(m^2·K) for the weighted average between central air and radiant temperature, named $\theta_{ref,i}$, as reference. Values for the convective part h_{ci} were given when discussing convection. For a surface facing the indoors, the radiant part is:

$$h_{ri} = 5.67 e_L F_T$$

with e_L the long wave emissivity of the surface and F_T the temperature ratio for radiation in the interval $\theta_{si} - \theta_{ri}$, mostly set equal to 0.95. Since most finishes have a long wave emissivity 0.8–0.9, that radiant part becomes:

$$4.3 \leq h_{ri} \leq 4.95 \ \ \mathrm{W/(m^2 \cdot K)}$$

As standard values for the (combined) inside surface film coefficient (W/(m^2·K)) are so advanced:

Vertical surfaces, slope > 45°	Horizontal surfaces, slope ≤ 45°
≈7.7	Heat flow up ($q\uparrow$) 10
	Heat flow down ($q\downarrow$) 6

The 2021 ASHRAE Handbook of Fundamentals gives a more complete set that takes into account the long wave emissivity of the surface:

Position	Heat flow direction	h_i (W/(m^2 · K)) for a surface emissivity		
		0.9	0.2	0.05
Horizontal	Upward	9.26	5.17	4.32
Sloping 45°	Upward	9.09	5.00	4.15
Vertical	Horizontal	8.29	4.20	3.35
Sloping 45°	Downward	7.50	3.41	2.56
Horizontal	Downward	6.13	2.10	1.25

None can be used for cases deviating substantially from the assumptions made. Then, the theory should help in fixing case-specific values.

How to value the reference temperature? Calculating the radiant temperature is quite complex, which is why, provided the room is beam shaped and all surfaces are grey with long wave emissivity ≈ 0.9, an area-weighted average of all surface temperatures figures as quite a good estimate:

$$\theta_{\mathrm{ri}} = \sum_{k=1}^{n} \left(A_k \theta_{\mathrm{sk}}\right) \Bigg/ \sum_{k=1}^{n} A_k \tag{1.100}$$

In case of grey vertical, sloped and horizontal envelope assemblies with the heat flowing up, the reference so becomes $\theta_{\mathrm{ref,\,i}} = 0.44\,\theta_{\mathrm{i}} + 0.56\,\theta_{\mathrm{ri}}$, a result close to the average between the central air and the radiant temperature. That average is called the operative temperature θ_{o}:

$$\theta_{\mathrm{ref,i}} = \theta_{\mathrm{o}} = \left(\theta_{\mathrm{i}} + \theta_{\mathrm{ri}}\right)/2 \tag{1.101}$$

For reflective surfaces, the operative temperature nears the central air temperature as convection then dominates, or $\theta_{\mathrm{o}} \approx \theta_{\mathrm{i}}$. For grey horizontal partitions and envelope assemblies with the heat flowing down, the reference temperature turns into:

$$\theta_{\mathrm{ref,i}} = 0.2\,\theta_{\mathrm{i}} + 0.8\,\theta_{\mathrm{ri}} \tag{1.102}$$

However, to note is that the larger the impact of envelope assemblies with really cold inside surfaces on the radiant temperature seen by vertical, sloped and horizontal inside partitions is, the less evident the use of the reference temperatures just defined becomes.

1.4.1.3 Outdoors

Three heat fluxes touch the outer surface of envelope assemblies. A first is convection between surface and the outside air:

$$q_{\mathrm{ce}} = h_{\mathrm{ce},j}(\overline{\theta}_{\mathrm{e},j} - \theta_{\mathrm{se}})$$

In it, $h_{\mathrm{ce},j}$ is the average convective surface film coefficient and $\overline{\theta}_{\mathrm{e},j}$ the average temperature of the air outside the boundary layer, usually assumed equal to the temperature measured by the nearest weather station under thermometer hut 1.7 m above grade (θ_{e}), or:

$$q_{\mathrm{ce}} = h_{\mathrm{ce}}(\theta_{\mathrm{e}} - \theta_{\mathrm{se}})$$

with h_{ce} the weather station linked convective surface film coefficient:

$$h_{\mathrm{ce}} = h_{\mathrm{ce},j}\frac{\overline{\theta}_{\mathrm{e},j} - \theta_{\mathrm{se}}}{\theta_{\mathrm{e}} - \theta_{\mathrm{se}}}$$

A second is long wave radiation between the surface, the terrestrial environment (e) and the sky (sk) seen as a black. The black emittance from the surface considered (s) to these two equals:

$$M_{\mathrm{bs}} = \left[1 + \frac{\rho_{\mathrm{Ls}}}{e_{\mathrm{Ls}}}(F_{\mathrm{se}} + F_{\mathrm{ssk}})\right] M'_{\mathrm{s}} - \frac{\rho_{\mathrm{Ls}}}{e_{\mathrm{Ls}}}\left(F_{\mathrm{se}} M'_{\mathrm{e}} + F_{\mathrm{ssk}} M_{\mathrm{bsk}}\right)$$

From the terrestrial environment (e) to the surface and to the sky the emittance in turn is:

$$M_{be} = \left[1 + \frac{\rho_{Le}}{e_{Le}}(F_{es} + F_{esk})\right] M'_e - \frac{\rho_{Le}}{e_{Le}} \left(F_{es}M'_s + F_{esk}M_{bsk}\right)$$

In both equations, e_{Ls} and ρ_{Ls} are the long wave emissivity and reflectivity of the surface, e_{Le} and ρ_{Le} the average long wave emissivity and reflectivity of the terrestrial environment, F_{se} the view factor between the surface and the terrestrial environment, F_{es} the view factor for the inverse, F_{ssk} the view factor between the surface and the sky, and F_{esk} the view factor between the terrestrial environment and the sky. M_{bsk} is the black body emittance of the sky while M'_s and M'_e are the radiosities of the surface and the terrestrial environment. As the other two surround the surface, $F_{se} + F_{ssk} = 1$ and due to the surface being infinitely small compared to the terrestrial environment, $F_{es} \approx 0$, while the view factor F_{esk} between the terrestrial environment and the sky ≈ 1 as nearly all its radiation goes to the sky. Both balances so simplify to:

$$M_{bs} = \frac{1}{e_{Ls}}M'_s - \frac{\rho_{Ls}}{e_{Ls}} \left(F_{se}M'_e + F_{ssk}M_{bsk}\right) \quad M_{be} = \frac{1}{e_{Le}} M'_e - \frac{\rho_{Le}}{e_{Le}}M_{bsk}$$

Solving both for M'_s and inserting the result in $q_{rse} + q_{rssk} = e_{Ls} \left(M_{bs} - M'_s\right) / r_{Ls}$ knowing that $e_{Ls}(F_{se} + F_{ssk})M_{bs} = e_{Ls}M_{bs}$ gives:

$$q_{rse} + q_{rssk} = q_{re} = e_{Ls}F_{se}(M_{bs} - e_{Le}M_{be}) + e_{Ls}F_{ssk}\left[M_{bs} - \left(\rho_{Le}\frac{F_{se}}{F_{ssk}} + 1\right)M_{bsk}\right]$$

Assumed now is that the terrestrial environment is black and at outside air temperature. The fact that during clear nights the sky temperature falls $\approx 21\,°C$ below the air temperature in the atmosphere, simplifies the radiant flux between surface and outside environment to:

$$q_{rs} = e_{Ls}C_b[(F_{se}F_{Tse} + F_{ssk}F_{Tssk})(\theta_e - \theta_{se}) - 21F_{ssk}F_{Tssk}(1 - 0.87c)]$$

In it, F_{Tse} is the temperature ratio for radiation between surface and terrestrial environment and F_{Tws} the temperature ratio between surface and sky. The factor c represents the cloudiness, 0 for a clear, 1 for an overcast sky.

A third is insolation. Although a daytime reality, the average impact on the heat received is large. Each m^2 of outside face absorbs the beam, diffuse and reflected radiation (E_{ST}) proportionally to its short-wave absorptivity (α_S):

$$q_{se} = \alpha_S E_{ST}$$

Summing up the convective, radiant and solar flux gives:

$$\begin{aligned} q_e = {} & h_{ce}(\theta_e - \theta_{se}) + 5.67\, e_{Ls}(F_{se}F_{Tse} + F_{ssk}F_{Tssk})(\theta_e - \theta_{se}) \\ & - 120\, e_{Ls}F_{ssk}F_{Tssk}(1 - f_c) + \alpha_K E_{ST} \end{aligned}$$

With $5.76\, e_{Ls}(F_{se}F_{Tse} + F_{ssk}F_{Tssk})$ the outside surface film coefficient $h_{e,r}$ for radiation, the equation rewrites as:

$$q_e = (h_{ce} + h_{re}) \left\{ \left[\theta_e + \frac{\alpha_K E_{ST} - e_{Ls}120\, F_{ssk}F_{Tssk}(1 - f_c)}{h_{ce} + h_{re}}\right] - \theta_{se} \right\} \qquad (1.103)$$

The term between [ ] has as units °C and is called the average sol-air temperature θ_e^* over a given period (one hour, one day, one week, one month). Its value depends on the radiant properties, the inclination and orientation of the surface, the weather and the period, all factors that differ between applications and ensure that the three fluxes discussed drive the heat exchange between outdoors and outer faces. With the sum $h_{ce} + h_{re}$ representing the outside surface film coefficient h_e, related flux becomes:

$$q_e = h_e \left(\theta_e^* - \theta_{se}\right) \tag{1.104}$$

As stated above, the convective part in h_e is 19 W/(m^2·K). For the radiant part, the temperature ratios for radiation F_{Tse} and F_{Tssk} equal to F_T and $F_{se} + F_{ssk} = 1$, gives:

$$h_{re} = 5.67 e_L F_T \tag{1.105}$$

As the temperature factor F_T could scatter between 0.8 and 0.9, the value is:

$$4.4\, e_L \leq h_{re} \leq 5.1\, e_L \text{ W/(m}^2\text{·K)}$$

Provided the outside surfaces have a long wave emissivity 0.9, 4–4.6 W/(m^2·K) looks likely. Or, the outside surface film coefficient for heat transfer will near 23 W/(m^2·K). The EN standard takes 25 W/(m^2·K), while the 2021 ASHRAE Handbook of Fundamentals differs between winter and summer:

	Direction of heat flow	h_e (W/(m^2·K))
Winter (wind speed 6.7 m/s)	Any	34.0
Summer (wind speed 3.4 m/s)	Any	22.7

A more accurate calculation should consider the whole heat balance, included a better guess of the average wind speed. Long-term measurements at the leeward side of an existing building for example gave on average a much lower outside surface film coefficient than 25 W/(m^2·K).

1.4.2 Steady-state, Flat Assemblies

1.4.2.1 Thermal Transmittance of Envelope Assemblies, Partitions and Party Walls

The use of surface film coefficients largely simplified the calculation of the steady-state heat flux ambient to ambient through flat assemblies. Take the outer wall of Figure 1.35.

Indoors, heating keeps the average reference temperature equal to θ_o °C. Outdoors, cold weather gives on average θ_e^* °C. From indoors to the wall's inner face, the heat flux now is:

$$q_1 = h_i(\theta_o - \theta_{si})$$

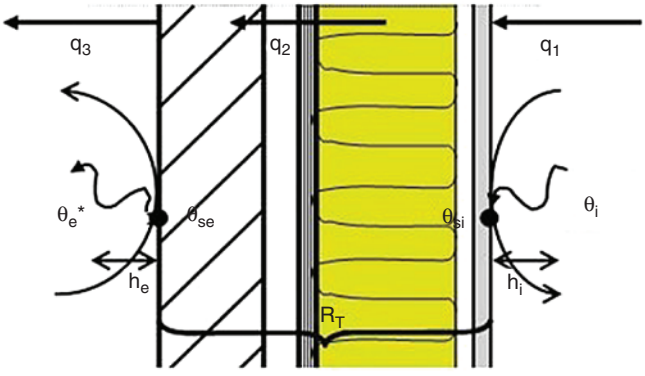


Figure 1.35 Outer wall, thermal transmittance.

with θ_{si} the inside surface temperature. Through the wall, the flux equals:

$$q_2 = (\theta_{si} - \theta_{se})/R_T$$

with θ_{se} the outside surface temperature and R_T the total thermal resistance of the wall. From the outer face to outdoors, the flux becomes:

$q_3 = h_e \left(\theta_{se} - \theta_e^*\right)$

In steady state, the three must be equal, common value q. Rearrangement and addition gives:

$$\begin{aligned} q/h_i &= \theta_o - \theta_{si} \\ +qR_T &= \theta_{si} - \theta_{se} \\ +q/h_e &= \theta_{se} - \theta_e^* \\ \hline \text{Sum}: \quad q\,(1/h_i + R_T + 1/h_e) &= \theta_o - \theta_e^* \end{aligned}$$

This result can be rewritten as $q = U\left(\theta_o - \theta_e^*\right)$ with:

$$U = \frac{1}{1/h_e + R + 1/h_i} \tag{1.106}$$

U is called the thermal transmittance of the outer wall, units W/(m^2·K). The lower its value, the less heat flows through, or, the property reflects the insulating quality of outer assemblies. The inverse is called the thermal resistance ambient to ambient, symbol R_a, units m^2·K/W. Calculating it looks simple: add the two surface resistances to the total resistance: indoors $1/h_i$, marked R_i, $= 0.13$ m^2·K/W for vertical surfaces, $= 0.1$ m^2·K/W for sloped and horizontal surfaces with the heat flowing up and $= 0.17$ m^2·K/W for horizontal surfaces with the heat flowing down, outdoors $1/h_e$, marked R_e, $= 0.04$ m^2·K/W. The thermal transmittance so defined is called 'clear wall', since two and three dimensional effects are not considered. The property is nonetheless used to characterize heterogeneous walls, take masonry, where the mortar joints and the vertically perforated bricks, see Figure 1.36, both with different λ-value, make the heat flow three-dimensionally.

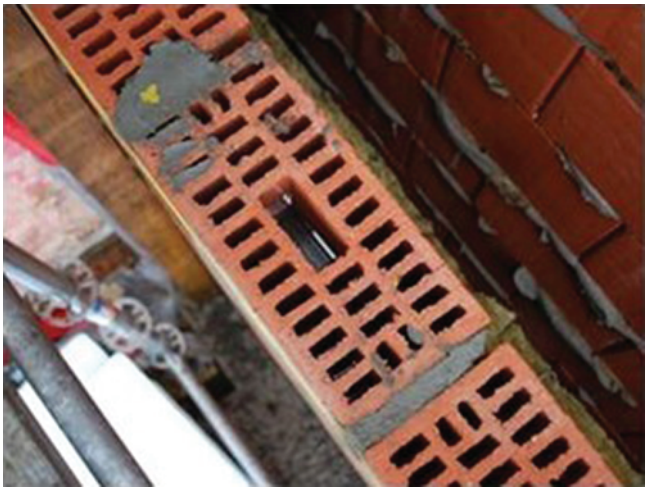


Figure 1.36 Masonry wall, heat transfer developing three-dimensionally.

For partitions and party walls, the surface film coefficients at both sides are those inside (h_i), giving as thermal transmittance:

$$U = 1/(R + 2/h_i) \tag{1.107}$$

The reference temperatures are the operative temperatures in the spaces the walls separate.

One remark anyhow. The U-value of multilayer assemblies as advanced does not consider contact resistances between layers. A main reason is these are so low that their impact is negligible. There is one exception anyhow: metal layers contacting each other. In such cases, possible contact resistances may have impact. The reason is clear: metals have such high thermal conductivity that the unintended enclosure of even thin air layers matters. This was the case with walls, consisting of cellular metal boxes filled with mineral wool and finished at the outside with a metal cladding, see Figure 1.37 for a wall with a thermal break between boxes and cladding.

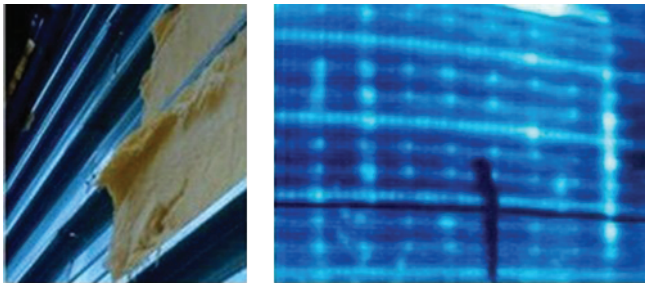


Figure 1.37 Sheet-metal assembly made of cellular metal boxes filled with mineral wool and finished with a metal cladding.

In theory, the clear wall thermal admittance should have been 0.36 W/(m²·K). The value measured without neither a thermal break nor an air layer left between the boxes and the cladding was 1.1 W/(m²·K), a direct consequence of thermal bridging where the box lips are fixed to and touch the cladding, see the figure right. Leaving an air layer between box and cladding with successive thicknesses of 0.5, 1.0 or 1.5 mm lowered the thermal transmittance to respectively 0.85, 0.78 and 0.74 W/(m²/K), still quite higher than 0.36 W/(m²·K).

1.4.2.2 Average Thermal Transmittance of Envelope Parts in Parallel

A facade with surface A_T consists of n different parts in parallel with surfaces A_i (Figure 1.38).

All face the outdoors. If lateral heat exchanges are negligible and if all see the same operative temperature indoors, the heat flow through each of them is:

$$\Phi_i = U_i A_i \Delta\theta$$

If facing different operative temperatures indoors ($\theta_{o,j}$) this equation converts to:

$$\Phi_i = a_i U_i A_i \Delta\theta$$

with 'a_i' a reduction factor equal to:

$$a_i = \frac{\theta_{o,j} - \theta_e}{\theta_{o,ref} - \theta_e}$$

In it, $\theta_{o,ref}$ is the operative temperature indoors taken as reference. The heat flow through the whole now must equal the sum of the heat flows through each part separately, or:

$$\Phi_T = \sum_{i=1}^{n} \Phi_i = \Delta\theta \sum_{i=1}^{n} (a_i U_i A_i) = \Delta\theta U_m \sum_{i=1}^{n} A_i \qquad (1.108)$$

with U_m the average clear wall thermal transmittance of the n parts connected in parallel, with a value equal to the sum of the products of all their weighted clear

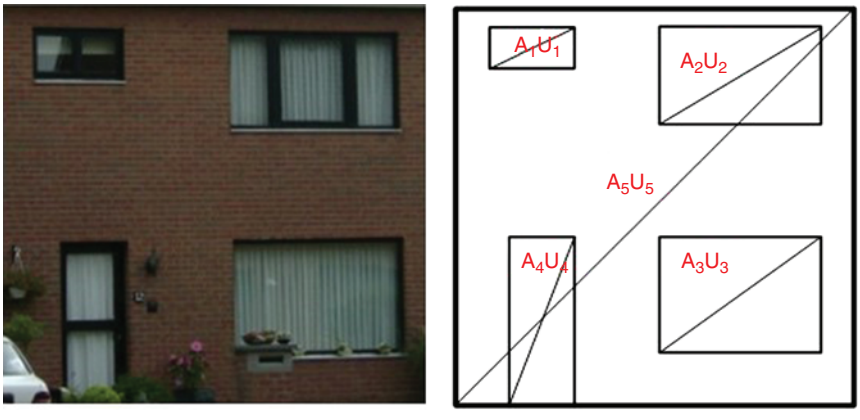


Figure 1.38 Assembly composed of parallel parts with hardly any lateral heat exchange.

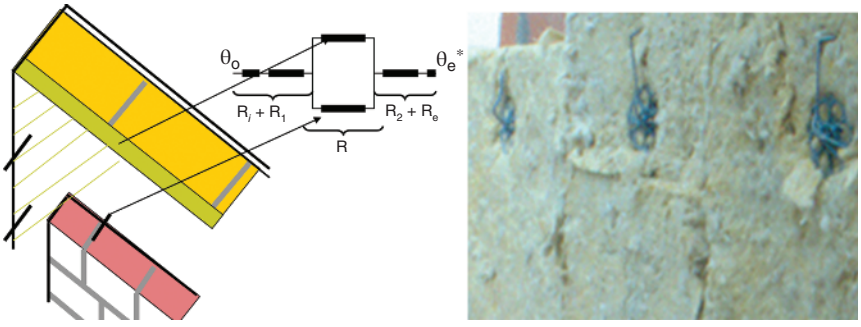


Figure 1.39 Cavity wall, electrical analogy accounting for the ties that perforate the fill.

wall thermal transmittances with their surface, divided by the whole surface:

$$U_{\mathrm{m}} = \sum_{i=1}^{n}(a_{\mathrm{i}}U_{\mathrm{i}}A_{\mathrm{i}}) \Big/ \sum_{i=1}^{n}A_{\mathrm{i}} = \sum_{i=1}^{n}(a_{\mathrm{i}}U_{\mathrm{i}}A_{\mathrm{i}}) \Big/ A_{\mathrm{T}} \tag{1.109}$$

Conversion to an overall thermal resistance gives:

$$R_{\mathrm{am}} = A_{\mathrm{T}} \Big/ \sum_{i=1}^{n}\left(\frac{a_{\mathrm{i}}A_{\mathrm{i}}}{R_{\mathrm{ai}}}\right) \tag{1.110}$$

1.4.2.3 Electrical Analogy

As long as lateral conduction between the composing parts is negligible, electric analogies allow solving quite complex cases. Take a cavity wall where the ties linking the veneer to the inside leaf perforate the cavity fill. Is A_{t} the whole tie section, R_{t} related thermal resistance, A_{is} the wall's surface and R_{is} the thermal resistance of the insulation, then the overall thermal resistance (R) of the insulation as perforated becomes (Figure 1.39):

$$R = \frac{A_{\mathrm{is}}}{\dfrac{A_{\mathrm{is}} - A_{\mathrm{t}}}{R_{\mathrm{is}}} + \dfrac{A_{\mathrm{t}}}{R_{\mathrm{t}}}}$$

With $R_1 + R_{\mathrm{i}}$ the thermal resistance from the insulation to the indoors and $R_2 + R_{\mathrm{e}}$ the thermal resistance from the insulation to the outdoors, the overall value in- to outdoors so becomes:

$$R_{\mathrm{T}} = (R_{\mathrm{i}} + R_1) + R + (R_2 + R_{\mathrm{e}}) = (R_{\mathrm{i}} + R_1) + \frac{A_{\mathrm{i}}}{\dfrac{A_{\mathrm{i}} - A_{\mathrm{t}}}{R_{\mathrm{is}}} + \dfrac{A_{\mathrm{t}}}{R_{\mathrm{t}}}} + (R_2 + R_{\mathrm{e}}) \tag{1.111}$$

1.4.2.4 Thermal Resistance of Non-ventilated Cavities

In an infinitely extending non-ventilated cavity, the distance and temperature difference between both end faces, their radiant properties, its slope, the heat flow direction and the mean temperature of the gas fill, all affect conduction, convection and radiation across, giving as heat flux:

$$q_{\mathrm{T}} = \left(\frac{\lambda_{\mathrm{g}}\,\mathrm{Nu}}{d} + \frac{C_{\mathrm{b}}\,F_{\mathrm{T}}}{1/e_{\mathrm{L1}} + 1/e_{\mathrm{L2}} - 1}\right)(\theta_{\mathrm{c1}} - \theta_{\mathrm{c2}})$$

with λ_g the thermal conductivity of the gas, Nu the case-specific Nusselt number, e_{L1} and e_{L2} the long wave emissivity and θ_{c1} and θ_{c2} the temperatures of both end faces. The thermal resistance so is:

$$R_c = \left(\frac{\lambda_g \, \mathrm{Nu}}{d} + \frac{C_b \, F_T}{1/e_{L1} + 1/e_{L2} - 1} \right)^{-1} \tag{1.112}$$

Figure 1.40 gives values for a vertical cavity filled with 10 °C warm air as function of thickness, temperature difference between and long wave emissivity of both end faces. That the thermal resistance increases when this long wave emissivity is lower and that, when low, the temperature difference between both end faces gains impact, underlines the dominance of radiation in the thermal resistance. The absence of any additional gain once the thickness passes 20–30 mm, in turn, underlines that radiation does not depend on the thickness of the cavity, while increased convection gradually compensates the drop in conduction through the gas.

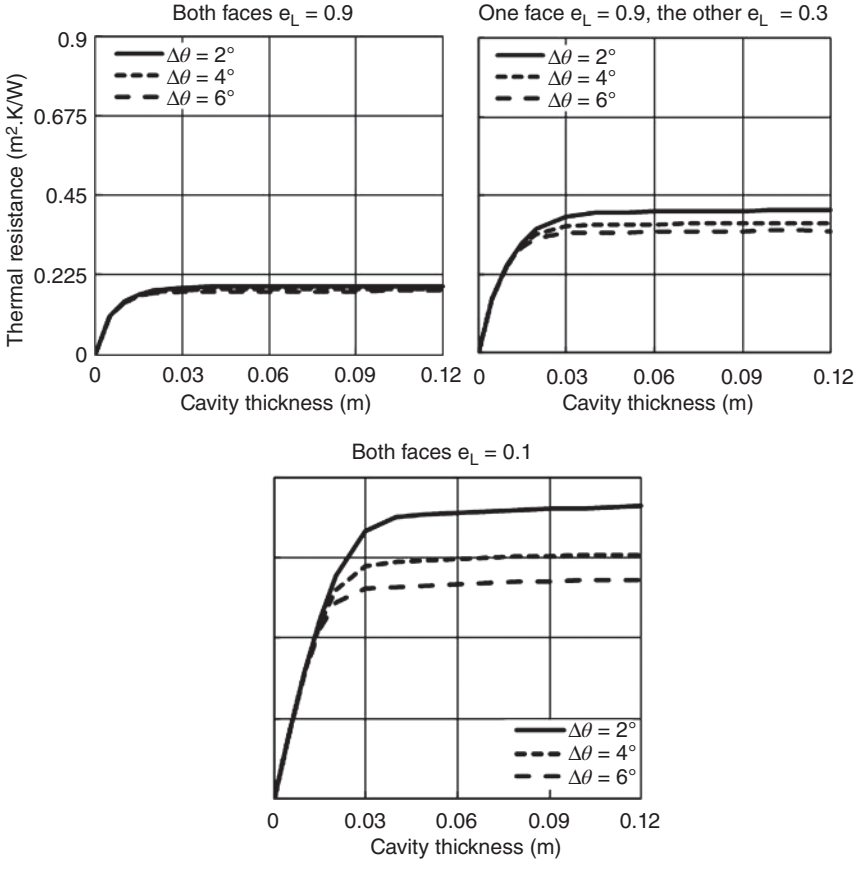


Figure 1.40 Thermal resistance of an infinite vertical cavity.

Table 1.3 Cavity, thermal resistance.

Thickness mm	Vertical cavity		Horizontal cavity	
	R_c (m² K/W), both surfaces grey	R_c (m² K/W), one surface reflecting	R_c (m² K/W) Heat flow up	R_c (m² K/W) Heat flow down
$0 < d < 5$	0.00		0.00	0.00
$5 \leq d < 7$	0.11		0.11	0.11
$7 \leq d < 10$	0.13		0.13	0.13
$10 \leq d < 15$	0.15		0.15	0.15
$15 \leq d < 25$	0.16	0.35	0.17	0.17
$25 \leq d < 50$	0.16	0.35	0.17	0.19
$50 \leq d < 100$	0.16	0.35	0.18	0.21
$100 \leq d < 300$	0.16	0.35	0.18	0.22
$d > 300$	0.16	0.35	0.18	0.23

For finite non-ventilated cavities, the values in Table 1.3 allow a first-order calculation.

1.4.2.5 Interface Temperatures

Considered is an envelope assembly. The surface resistances R_i and R_e are assumed representing the thermal resistance of a 1 m thick air layer with thermal conductivity h_i or h_e, added to the inner and outer face with as temperature differences over them the one between the reference in- ($R = R_a$, $\theta = \theta_i$) and outside ($R = 0$, $\theta = \theta_e^*$) and the temperature on these faces.

For multi-layer envelope assemblies, the temperature course in a $[R, \theta]$ axis system then is a straight line linking $\left[0, \theta_e^*\right]$ to $[R_a, \theta_i]$ with as slope the heat flux. Idem so for party walls and partitions, then linking $[0, \theta_{i,1}]$ to $[R_a, \theta_{i,2}]$. Of course, when tracing this line, the layer sequence must be respected, see Figure 1.41. The temperature on the inner face so is:

$$\text{Envelope assembly: } \theta_{si} = \theta_o - R_i \frac{\theta_o - \theta_e^*}{R_a} = \theta_o - \frac{U_{h_i}}{h_i}\left(\theta_o - \theta_e^*\right) \tag{1.113}$$

$$\text{Partitions and party walls: } \theta_{si} = \theta_{i.1} - R_i \frac{\theta_{o,1} - \theta_{o,2}}{R_a} = \theta_{o,1} - \frac{U_{h_i}}{h_i}(\theta_{o,1} - \theta_{o,2}) \tag{1.114}$$

The suffix $_{h_i}$ underlines that the clear wall thermal transmittance in the ratio with the surface film coefficient as denominator must be calculated using the same value as used for that surface film coefficient. The temperatures in the interfaces in turn equal:

$$\theta_x = \theta_i - q\left(R_i + R_{si}^x\right)$$

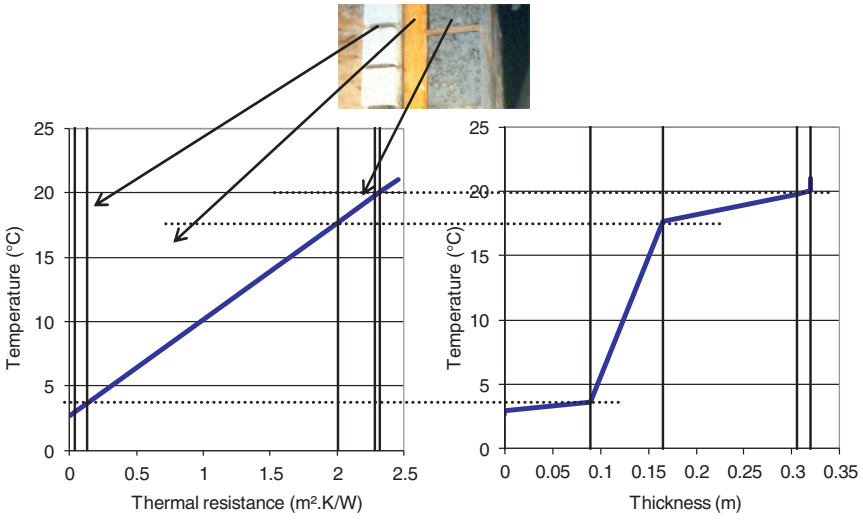


Figure 1.41 Composite envelope assembly (filled cavity wall): temperature line.

Transposing the straight line from this $[R, \theta]$ to the $[d, \theta]$ axis system goes as explained when discussing conduction through multi-layer flat assemblies, see Figure 1.41.

A same approach applies to party walls and partitions, be it with a surface resistance R_i at both faces and as reference the operative temperatures θ_i in both spaces.

1.4.2.6 Effect of Ever Thicker Insulation Layers on the Thermal Transmittance

The formula for the thermal transmittance of a multi-layer envelope assembly learns that a thicker thermal insulation pushes its value down, which, if done for all envelope parts, should result in less energy used for heating, as needed looking to the pursuit of zero carbon buildings by 2050. Question however is what happens with that less when an ever-thicker insulation layer is built-in? The answer is shown by the U-value as function of the insulation thickness and by its derivative to the insulation thickness:

$$\frac{\mathrm{d}U}{\mathrm{d}d_{\mathrm{ins}}} = \frac{d}{d_{\mathrm{ins}}}\left(\frac{1}{R_{\mathrm{o}} + {}^{d_{\mathrm{ins}}}/_{\lambda_{\mathrm{ins}}}}\right) = \frac{-1/\lambda_{\mathrm{ins}}}{\left(R_{\mathrm{o}} + {}^{d_{\mathrm{ins}}}/_{\lambda_{\mathrm{ins}}}\right)^2}$$

In it, R_o is the thermal resistance of all layers included the surface film resistances but without the insulation layer, while d_{ins} is the thickness in m and λ_{ins} the thermal conductivity in W/(m·K) of that insulation layer.

To clarify the result, considered again is the cavity wall of Figure 1.41. The thermal resistance ambient to ambient excluded the cavity fill is 0.34 m²·K/W. The glass wool insulation used as fill has as thermal conductivity 0.04 W/(m·K). Increasing its thickness stepwise from 4 to 40 cm gives a hyperbolic decrease of the U-value, while the derivative to this thickness underlines how rapidly the fall (−) goes, see Figure 1.42, which shows that the extra gains by lowering the U-value diminish quickly with increasing insulation thickness, meaning that a stepwise lowering of the values mandated brings ever less additional gain, so, has its limits.

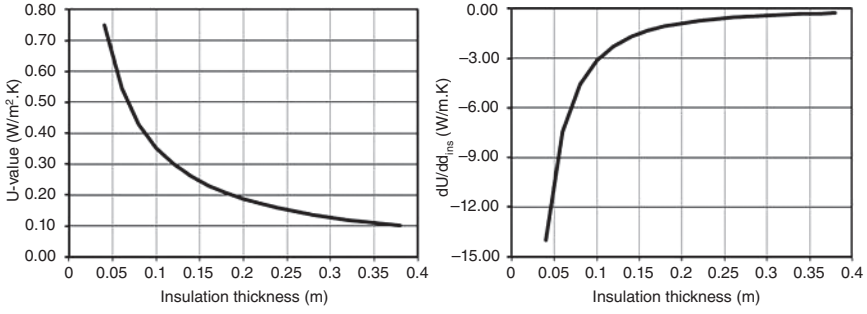


Figure 1.42 Impact of the insulation thickness on the thermal transmittance of the cavity wall of Figure 1.40.

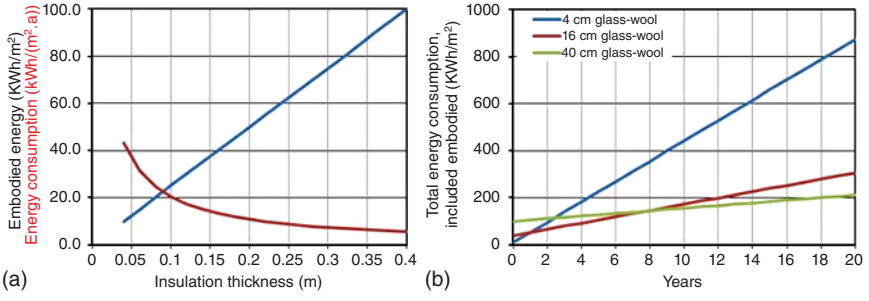


Figure 1.43 (a) Embodied energy in 1 m^2 of the glass wool, compared to the annual energy use for heating, (b) the total over a period of 20 years, included the embodied part.

Producing glass wool of course also requires energy, called the embodied energy. The thicker the insulation layer, the higher the embodied, see Figure 1.43. The figure shows that the total heating energy used per 1 m^2 of wall over a period of 20 years, included the embodied in the insulation is represented by the jump in energy used at year zero. While with 4 cm glass wool, the embodied part hardly matters, with 16 cm, giving a U-value 0.23 W/(m^2·K), it takes one year to see the energy needed dropping below the 4 cm case. For 40 cm, giving a U-value 0.10 W/(m^2.K), less energy needed compared to 4 cm demands some 3 years to drop below but compared to 16 cm up to 8 years are needed to drop below. There is even more. With 40 cm insulation in the cavity, the wall thickness turns ≈65 cm instead of the 30 cm with 4 cm in the cavity. In case an equal usable floor surface is requested, this 40 cm demands more built-up floor surface, or, in case built-up must remain the same, gives less usable floor surface. All this does not counteract the importance of an excellent thermal insulation, though without exaggerating how low the thermal transmittances mandated must be.

1.4.2.7 Solar Transmittance

The solar heat flux through an envelope part can be written as:

$$q_S = gE_{ST} \tag{1.115}$$

with E_{ST} the incident solar radiation on the outer face and q_S the heat flux transmitted, both in W/m^2. The quantity g, called the solar transmittance, encompasses the direct and indirect solar gains. The direct are:

$$q_{Sd} = \tau_S E_{ST}$$

with τ_S the total shortwave transmissivity of the part. For opaque parts, τ_S and the direct gains are zero. Not so for transparent parts. Indirect gains occur because opaque and transparent parts absorb a fraction of the incident solar heat. That warming conducts part of it to the inside face, where convection and long wave radiation dissipates it into the indoor ambient.

For single glass with shortwave absorptivity α_S, this resembles a surface-linked heat exchange with the indoors, equal to:

$$q_{Si} = h_i(\theta_{si} - \theta_o) \tag{1.116}$$

where θ_{si} is the unknown solar induced inside surface temperature. As such single pane warms up nearly homogeneously, its temperature remains close to constant over the glazing's thickness, giving as heat balance per m^2

$$\alpha_S E_{ST} + h_e(\theta_e - \theta_x) + h_i(\theta_o - \theta_x) = 0$$

In it, θ_e is the air temperature outdoors, θ_o the operative temperature indoors and θ_x the glass temperature with as value:

$$\theta_x = \theta_{si} = \frac{\alpha_S E_{ST}}{h_i + h_e} + \frac{h_e\theta_e + h_i\theta_i}{h_i + h_e}$$

The second term on the right side in this equation stands for the temperature the glass should have without sun, and the first term for the increase due to the absorbed solar heat. Combination with the equation for the heat flux to the indoors gives:

$$q_{Si} = \frac{h_i\alpha_S E_{ST}}{h_i + h_e} + \frac{h_i h_e(\theta_e - \theta_i)}{h_i + h_e} \tag{1.117}$$

Only the first term on the right side is sun-linked, so represents the indirect gains:

$$q_{Si} = \frac{h_i\alpha_S E_{ST}}{h_i + h_e}$$

The solar transmittance for single glass so becomes:

$$g = \frac{q_{Sd} + q_{Si}}{E_{ST}} = \tau_S + \frac{\alpha_S}{1 + h_e/h_i} \tag{1.118}$$

Or, the gains not only depend on the short-wave transmissivity of the glass but also on its short-wave absorptivity and the ease with which the absorbed heat is dissipated to the indoors.

For double glass, calculating the solar transmittance is more demanding. Let τ_{S1}, ρ_{S1}, α_{S1} and τ_{S2}, ρ_{S2}, α_{S1} be the transmissivity, reflectivity and absorptivity, all short-wave, of the one and the other pane. Reflection in the cavity breaks the solar radiation transmitted into a geometric series of added transmissions with ratio $\rho_{S1}\rho_{S2}$, having as sum (Figure 1.44):

$$q_{Sd} = \frac{\tau_{S1}\tau_{S2}}{1 - \rho_{S1}\rho_{S2}} E_{ST} \tag{1.119}$$

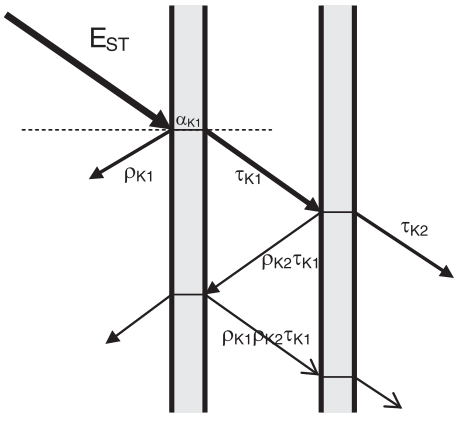


Figure 1.44 Double glass, solar transmittance.

As the denominator $1-\rho_{S1}\rho_{S2}$ is ≈ 1, it's the product of the transmissivities of both panes that mainly fixes the transmission through double glass.

For $\theta_e = \theta_o = 0\,°C$ and both panes isothermal, the indirect gains become $q_{Si} = h_i\theta_{x2}$ with θ_{x2} the temperature of the sun shone inner pane, a value ensuing from the heat balances per pane (1 is the outer, 2 the inner):

$$Pane\,1\colon\ \alpha_{S1}\frac{1-\rho_{s1}\rho_{s2}+\tau_{s1}\rho_{s2}}{1-\rho_{s1}\rho_{s2}}E_{ST}-h_e\theta_{s1}+\frac{\theta_{x2}-\theta_{x1}}{R_c}=0$$

$$Pane\,2\colon\ \alpha_{S2}\frac{\tau_{s1}}{1-\rho_{s1}\rho_{s2}}E_{ST}+\frac{\theta_{x1}-\theta_{x2}}{R_c}-h_i\theta_{x2}=0$$

Solving gives:

$$\theta_{x2}=\frac{\dfrac{\alpha_{S1}f_1}{R_c}+\alpha_{S2}f_2\left(h_e+\dfrac{1}{R_c}\right)}{\left(h_i+\dfrac{1}{R_c}\right)\left(h_e+\dfrac{1}{R_c}\right)-\dfrac{1}{R_c^2}}E_{ST} \tag{1.120}$$

Inserting this value in the equation for the indirect gains and adding the direct gives as solar transmissivity of double glazing:

$$g=\tau_{S1}\tau_{S2}+h_i\frac{\dfrac{\alpha_{S1}f_1}{R_c}+\alpha_{S2}f_2\left(h_e+\dfrac{1}{R_c}\right)}{\left(h_i+\dfrac{1}{R_c}\right)\left(h_e+\dfrac{1}{R_c}\right)-\dfrac{1}{R_c^2}} \tag{1.121}$$

This result shows how to decrease it: limit the direct transmission and lower either the inside surface film coefficient or the shortwave absorptivity of the panes.

For multiple glazing and the combination of glass and solar shading, the same approach applies: the transmittance for the direct gains equal to the product of the shortwave transmissivities of the panes and the shading, for the indirect gains equal to the result of solving the system of heat balances per pane plus the shading.

1.4.3 Local Inside Surface Film Coefficients

The surface film coefficients as given are of help in quantifying the area averaged heat losses and gains. To get the surface temperatures or heat fluxes at specific spots, local values should be used. As start holds:

$$h_{ix}(\theta_{ref,i} - \theta_{six}) = h_{cix}(\theta_{ix} - \theta_{six}) + h_{rix}(\theta_{rix} - \theta_{six}) \tag{1.122}$$

with h_{ix} the local surface film coefficient linked to a reference temperature $\theta_{ref,i}$, h_{cix} the local convective surface film coefficient linked to the local air temperature θ_{ix} outside the boundary layer, h_{rix} the local radiant surface film coefficient linked to the radiant temperature θ_{rix} seen by the spot considered and θ_{six} the local surface temperature. If R' is the equivalent thermal resistance linking the inside face at that spot to the ambient at the other side, then:

$$h_{ix}(\theta_{ref.,i} - \theta_{six}) = (\theta_{six} - \theta_j)/R' \tag{1.123}$$

This equation is an approximation because in reality, such equivalent thermal resistance depends on how the local inside surface film coefficients (h_{ix}) are distributed over the inner face. For envelopes, θ_j is the temperature outdoors ($j = e$), for partitions and party walls the reference temperature in the adjacent space. Eliminating the local surface temperature θ_{six} from the two equations above and solving for the local surface film coefficient gives:

$$h_{ix} = \frac{h_{cix} + h_{rix} - p_T}{1 + R' p_T} \quad \text{with} \quad p_T = \frac{h_{cix}(\theta_{ref.,i} - \theta_{ix}) + h_{rix}(\theta_{ref.,i} - \theta_{rix})}{\theta_{ref,i} - \theta_j} \tag{1.124}$$

In case the reference temperature indoors ($\theta_{ref,i}$), its relation with the air temperature just outside the boundary layer (θ_{ix}), its relation with the radiant temperature seen by the spot (θ_{rix}) are known and the local inside surface film coefficients h_{cix} and h_{rix} are quantified, then, on condition the equivalent thermal resistance is known, the equations above allow quantifying the heat flux at the spot considered.

Questions anyhow left are: how to link the local reference temperature indoors to the overall reference indoors ($\theta_{ref,i}$), values of the local surface film coefficients? Taken as overall reference indoors is the air temperature θ_i in the room's centre, 1.7 m above floor level. Assuming the air temperature there (θ_{ix}) increases linearly along the vertical, be it less when the enclosure is better insulated and heating less convective, the relation with the overall reference could be:

$$\frac{\theta_{ix} - \theta_j}{\theta_i - \theta_j} = 1 + 0.2\, p_c U_m (y - 1.7) \tag{1.125}$$

with y a spot on that vertical, θ_j the reference temperature in the adjacent space or outdoors, p_c a convection factor (1 for air heating, 0.9 for convector heating, 0.4–0.8 for radiator heating, 0.4 for floor heating), and U_m the average thermal transmittance of all walls enclosing the space considered. The relation comes from data gathered in a test room with varying enclosure insulation and warmed with several heating

systems. Computer simulations of the radiant heat exchange in spaces with different shape on the other hand showed that the local radiant temperature (θ_{rix}) could be given a value proportional to the reference, with gradient depending on the local convective surface film coefficient, the convection factor and the average U_m-value:

$$\frac{\theta_{rix} - \theta_i}{\theta_i - \theta_j} = \frac{h_{cix}}{h_{cix} + \frac{(p_c - 0.4)\, U_m}{0.6}} \tag{1.126}$$

Further on, the local convective surface film coefficient (h_{cix}) could be assumed being 2.5 W/(m^2·K), while following values can be advanced for the radiant surface film coefficient (h_{rix}):

Corner between three envelope assemblies or two and a partition	
Spot more than 0.5 m from an edge	5.5 e_L
Spot less than 0.5 m from an edge but more than 0.5 m from a corner	3.4 e_L
Spot less than 0.5 m from a corner	2.2 e_L
Edge between two envelope assemblies or one and a partition	
Spot more than 0.5 m from an edge	5.5 e_L
Spot less than 0.5 m from an edge	3.4 e_L
Envelope assembly or partition	
Spot anywhere	5.5 e_L

Where furniture hides a wall, a combined inside surface film coefficient of 2 W/(m^2 · K) is used.

1.4.4 Steady-state: Two and Three Dimensions

1.4.4.1 Pipes

The heat flow between the pipe's outer face and the ambient all-around is (Figure 1.45):

$$\Phi_{n+1} = 2\pi R_{n+1} h_2 (\theta_{s,2} - \theta_{ref,2}) \tag{1.127}$$

with h_2 the surface film coefficient, $\theta_{ref,2}$ the reference temperature in the ambient, $\theta_{s,2}$ temperature on the pipe's outside face and R_{n+1} the radius between the pipe's centre and that face.

At the pipe's inner face, the heat flow is:

$$\Phi_1 = 2\pi R_1 h_1 (\theta_{ref,1} - \theta_{s,1}) \tag{1.128}$$

with h_1 the surface film coefficient and $\theta_{ref,1}$ the temperature of the fluid in the pipe, $\theta_{s,1}$ related surface temperature and R_1 the pipe's inner radius.

Through the pipe, the flow becomes:

$$\Phi_{1,n+1} = \frac{\theta_{s,1} - \theta_{s,2}}{\sum_{i=1}^{n} \left[\frac{\ln(R_{i+1}/R_i)}{2\pi \lambda_i} \right]} \tag{1.129}$$

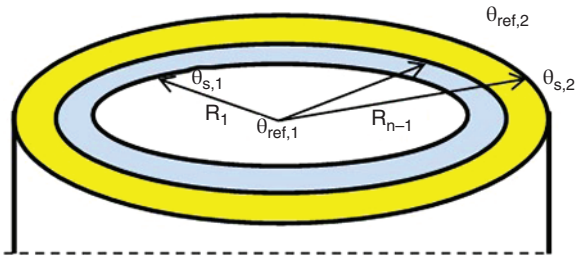


Figure 1.45 The pipe problem.

Σ because the pipe may be composed of several layers. In steady state, the three heat flows must be equal. Reshuffling and adding so gives:

$$\frac{\Phi_{n+1}}{2\pi R_{n+1}h_2} = \theta_{s,2} - \theta_{ref,2}$$

$$+\Phi_{1,n+1}\sum_{i=1}^{n}\left[\frac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] = \theta_{s,1} - \theta_{s,2}$$

$$+\frac{\Phi_1}{\theta_{ref,1} - \theta_{s,1}} = 2\pi R_1 h_1$$

$$\Phi_{1,n+1}\left\{\frac{1}{2\pi R_{n+1}h_2} + \sum_{i=1}^{n}\left[\frac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] + \frac{1}{2\pi R_1 h_1}\right\} = \theta_{ref,1} - \theta_{ref,2}$$

As for flat assemblies, this sum can be rewritten as:

$$\Phi_{1,n+1} = U_{pipe}(\theta_{ref,1} - \theta_{ref,2}) \tag{1.130}$$

where U_{pipe} is the thermal transmittance per meter run of the pipe:

$$U_{pipe} = \frac{1}{\frac{1}{2\pi R_{n+1}h_2} + \sum_{i=1}^{n}\left[\frac{\ln(R_{i+1}/R_i)}{2\pi\lambda_i}\right] + \frac{1}{2\pi R_1 h_1}} \quad (W/(m{\cdot}K)) \tag{1.131}$$

Insulating a pipe lowers its heat loss or gain, depending on whether the fluid transported is warm or cold. The benefit of thicker insulation however drops faster than for flat assemblies.

1.4.4.2 Floors on Grade

The heat loss of a floor on grade to the soil is a three-dimensional reality. Although the concept 'thermal transmittance' does not fit in such case, a simplified flat surface approach, using an equivalent thermal transmittance, is used. For that, the thermal transmittance is rewritten as:

$$U = aU_{o,floor} \tag{1.132}$$

with 'a' a reduction factor and $U_{o,floor}$ the thermal transmittance of the floor as if flat and facing the outdoors. Calculating 'a' starts with fixing the characteristic floor dimension:

$$B' = 2A_{fl}/P \ \text{(m)} \tag{1.133}$$

Figure 1.46 Arrow showing a fraction of the free perimeter.

In it, A_{fl} is the floor surface and P that part of the floor perimeter contacting the outdoors and called the free perimeter, see Figure 1.46.

Next, the equivalent soil thickness (d_t) of the floor is calculated as:

$$d_t = d_{fw} + \lambda_{gr}\left(\frac{1}{h_e} + R_{T,fl} + \frac{1}{h_i}\right) \text{ (m)} \tag{1.134}$$

with d_{fw} the average thickness of the foundation walls along the free perimeter in m, λ_{gr} the thermal conductivity of the soil, $R_{T,fl}$ the thermal resistance of the floor, h_i the surface film coefficient indoors, 6 W/(m^2·K)), and h_e the surface film coefficient outdoors, 25 W/(m^2·K). The reduction factor a finally depends on how the equivalent soil thickness compares to B':

If $d_t < B'$, then $a = \frac{1}{U_{o,floor}}\left(\frac{2\lambda_{gr}}{\pi B' + d_t}\right)\ln\left(\frac{\pi B'}{d_t} + 1\right)$ If $d_t \geq B'$, then $a = \frac{1}{U_{o,floor}}\left(\frac{\lambda_{gr}}{0.457B' + d_t}\right)$

1.4.4.3 Thermal Bridges

A thermal bridge means not only more heat loss or gain than through the surrounding flat parts, except if single or double glazing, but also the inside face there colder during the heating season. To calculate the isoflux lines and the isotherms, CVM is used, taking into account that the heat always enters a thermal bridge perpendicular to the inner face and leaves it perpendicular to the outer face. Calculating surface temperatures require the use of local, the heat flows the use of the standard surface film resistances. The energy balance for a control volume with its centre on an in- or on an outer face so combines six heat flows: four from the four adjacent centres on that face, one from the adjacent control volume in the material layer at that face and one perpendicular to the face from the ambient at reference temperature through the surface film resistance, see Figure 1.47.

The control volumes in the material touching an end face extending parallel to the $[y, z]$-plane have a surface a^2 and a depth $a/2$. The heat coming from the ambient at temperature $\theta_{i,m,n}$ and flowing to one of these, having as surface temperature $\theta_{s,m,n}$, equals:

$$\Phi_{s,m,n}^{i,m,n} = h_i(\theta_{i,m,n} - \theta_{s,m,n})a^2$$

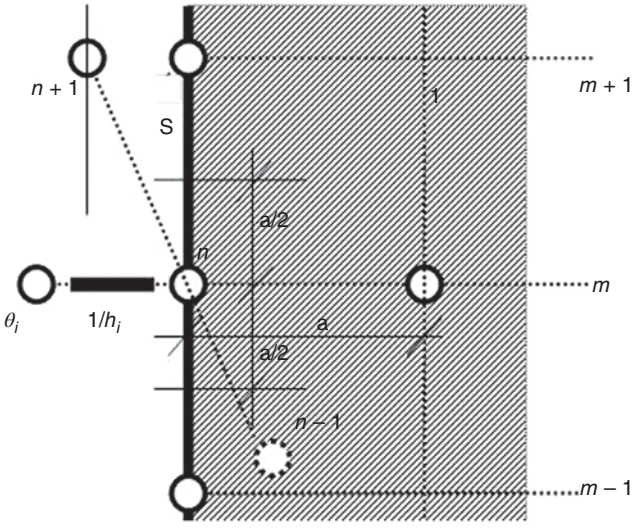


Figure 1.47 CVM-method, control volumes at the inside or outside surface.

The heat flows from the adjacent four at the end face crossing the distance a over a surface $a^2/2$ are

$$\Phi^{s,m,n}_{s,m+1,n} = \lambda_1(\theta_{s,m+1,n} - \theta_{s,m,n})\,\frac{a}{2} \qquad \Phi^{s,m,n}_{s,m-1,n} = \lambda_1(\theta_{s,m-1,n} - \theta_{s,m,n})\frac{a}{2}$$

$$\Phi^{s,m,n}_{s,m,n+1} = \lambda_1(\theta_{s,m,n+1} - \theta_{s,m,n})\frac{a}{2} \qquad \Phi^{s,m,n}_{s,m,n-1} = \lambda_1(\theta_{s,m,n-1} - \theta_{s,m,n})\frac{a}{2}$$

The heat flow from the adjacent control volume in the material with centre at distance a from the end face to the central on the end face in turn writes:

$$\Phi^{s,m,n}_{l,m,n} = \lambda_1(\theta_{l,m,n} - \theta_{s,m,n})a$$

Sum zero gives:

$$ah_i\theta_{i,m,n} + \lambda_1\frac{\theta_{s,m+1,n} + \theta_{s,m-1,n} + \theta_{s,m,n+1} + \theta_{s,m,n-1}}{2} + \lambda_1\theta_{l,m,n} - (ah_i + 3\lambda_1)\theta_{s,l,m,n} = 0$$

with $ah_i\theta_{i,\,m,\,n}$ known. Central control volumes at corners give analogous equations.

A difference now is made between geometrical and structural thermal bridges (Figure 1.48).

The geometrical are linked to the form building enclosures have. Structural ones follow from structural decisions: concrete beams and columns contacting the outdoors, discontinuities in the thermal insulation, etc. Often the need for structural integrity is a cause. Take a balcony. The cantilever moment requires continuity with the floor slab, thus a thermal bridge. Excluding requires continuity of the thermal insulation. But complete continuity cannot, although the higher heat flows and lower surface temperatures the continuity needed induces should remain acceptable.

To facilitate calculating the heat losses and gains due to thermal bridging, the concept of linear and local thermal transmittances has been introduced. The first,

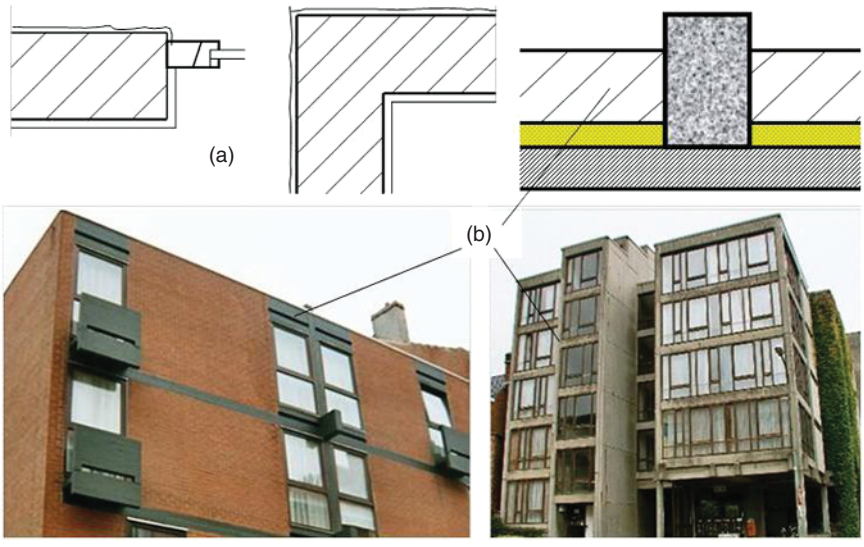


Figure 1.48 (a) Geometric thermal bridges, (b) structural thermal bridges.

symbol ψ, units W/(m·K), quantify the extra heat flow a two-dimensional thermal bridge induces per running metre and 1 K temperature difference between the ambient at both sides compared to no thermal bridge being there. The second, symbol χ, units W/K, does the same for a three-dimensional thermal bridge.

Calculating linear thermal transmittances demands a well-defined clear wall reference and agreement on which end face to consider, in- or outside. 'Outside' is preferred as it allows using facade drawings. First, the detail forming the thermal bridge is omitted and the one-dimensional clear wall heat flow calculated. Then, the linear thermal bridge using the correct drawings is added, after which CVM-meshing allows quantifying the two-dimensional heat flow and the inside surface temperatures (Figure 1.49). Are Φ_{2D} and Φ_o the heat flows through the wall with and without thermal bridge, then the linear thermal transmittance with L its length equals:

$$\psi = \frac{\Phi_{2D} - \Phi_o}{L\,\Delta\theta} \tag{1.135}$$

If an assembly contains a local thermal bridge, the local thermal transmittance (χ) so becomes:

$$\psi = \frac{\Phi_{3D} - \Phi_o}{L\,\Delta\theta} \tag{1.136}$$

Often local thermal bridges emerge where linear cross each other. If so, three cases must be calculated: first the clear wall, then the wall with consecutively each of the linear thermal bridges and finally the wall three-dimensionally. After, the local thermal transmittance is extracted as:

$$\chi = \frac{\Phi_{3D} - \Phi_{2D}}{\Delta\theta} \tag{1.137}$$

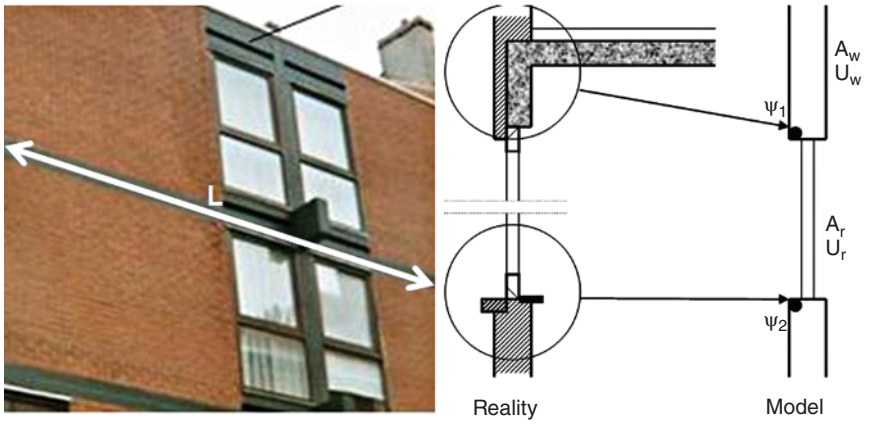


Figure 1.49 Linear thermal transmittances. The dummy consists of flat parts with lines, in cross-section points, that replace the linear thermal bridges.

Once all linear and local thermal transmittances are quantified, the whole wall thermal transmittance of a flat assembly containing thermal bridges follows from:

$$U = U_o + \frac{\sum_{i=1}^{n}(\psi_i L_i) + \sum_{j=1}^{m}\chi_j}{A} \tag{1.138}$$

with U_o the clear wall thermal transmittance, A the surface considered, n the number of linear thermal bridges over that surface, L_i their length and m the number of local thermal bridges over it.

A two or three dimensional calculation with the local inside surface film coefficient (see above) in turn will give the lowest inside surface temperature $\theta_{s,min}$, which allows characterizing the impact of thermal bridges on the inside surface temperature by using a non-dimensional temperature factor:

$$f_{h_i} = \frac{\theta_{s,min} - \theta_e}{\theta_o - \theta_e} \tag{1.139}$$

where θ_o and θ_e are the reference temperatures in- and outdoors. The suffix h_i reminds that the local surface film coefficient must be used. A CVM calculation with 1 K temperature difference between the ambient at both sides directly gives that temperature factor.

The higher the linear or local thermal transmittance and the lower the temperature factor, the more problematic a thermal bridge. While often taking a disproportionate share in the heat losses or gains by degrading insulation efficiency, their inside surfaces may collect more dirt, may see mould growth risk increase and may turn into preferred spots for surface condensation and crack formation.

Available is software that allows calculating the two and three dimensional heat flows through, the isotherms plus isoflux lines across and related linear or local thermal transmittances and temperature factors for any thermal bridge, requiring as input the geometry, the meshing and the material properties of the composing layers, see Figure 1.50 for a result.

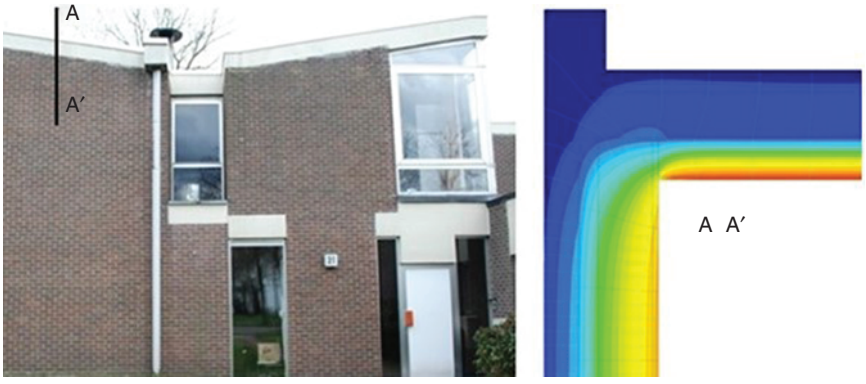


Figure 1.50 Roof edge as thermal bridge, the isotherms for 20 °C in- and 0 °C outdoors, calculated using appropriate software.

1.4.4.4 Windows

Windows transfer heat three-dimensionally as the IR image of Figure 1.51 shows. Quantifying their thermal transmittance (U_{window}) so requires appropriate software. However, as using this software for any case encountered could be far too time-consuming, frames are characterized by an equivalent thermal transmittance ($U_{\text{eq,frame}}$), multi-pane glasses by their central thermal transmittance ($U_{\text{o,glass}}$) and the spacer/frame combinations, see the white rectangle in Figure 1.51, by a linear thermal transmittance (ψ_{spacer}). This allows writing U_{window} as:

$$U_{\text{window}} = \frac{A_{\text{glass}} U_{\text{o,glass}} + A_{\text{frame}} U_{\text{eq,frame}} + \psi_{\text{spacer}} L_{\text{spacer}}}{A_{\text{window}}} \tag{1.140}$$

The frame surface (A_{frame}) is assumed equal to its normal projection on a plane parallel to the window's outer face. The visible glass surface (A_{glass}) is fixed the same way, while the length of the spacers (L_{spacer}) follows from the sum of the perimeters of

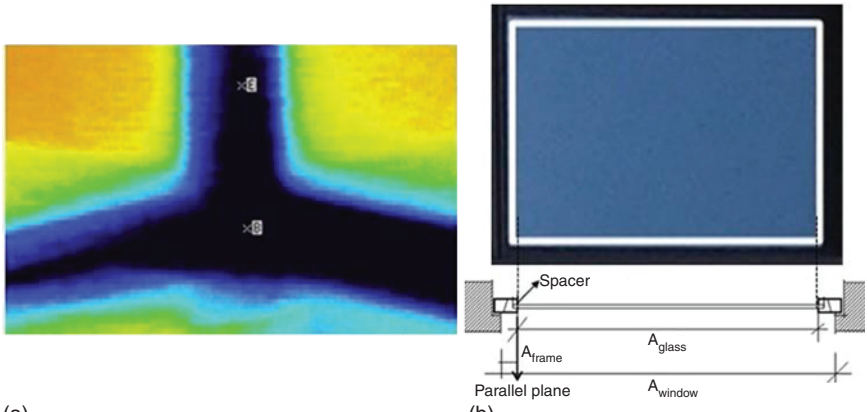


Figure 1.51 Window, (a) an IR-image of the frame containing double glazing, (b) calculating related thermal transmittance.

Table 1.4 Frames, glass and spacers, thermal transmittances and linear thermal transmittances.

Frames	$U_{eq,frame}$ (W/(m²·K))	Glazing	$U_{o,glass}$ (W/(m²·K))
Hardwood, $d = 70$ mm	2.08	Double	2.8
Aluminium, 20 mm thermal cut	2.75	Double, low-e, argon-filled	1.1
PVC, three air holes frame	2.00	Triple, low-e, argon filled	0.6
	Spacers		
Metal	ψ_{spacer} W/(m·K)	Insulating	ψ_{spacer} W/(m·K)
$U_{eq,frame} < 5.9$ W/(m²·K) $U_{o,glass} > 2.0$ W/(m²·K)	0.06	$U_{eq,frame} < 5.9$ W/(m²·K) $U_{o,glass} > 2.0$ W/(m²·K)	0.05
$U_{eq,frame} < 5.9$ W/(m²·K) $U_{o,glass} < 2.0$ W/(m²·K)	0.11	$U_{eq,frame} < 5.9$ W/(m²·K) $U_{o,glass} < 2.0$ W/(m²·K)	0.07

all separate glass parts. Table 1.4 lists practical values for the thermal transmittances and linear thermal transmittances of different frames, glazing and spacers.

1.4.4.5 Building Envelopes

Building envelopes, also called enclosures, have as functions protecting the indoors from the weather and shielding it from the soil, from crawlspaces, unheated basements, sometimes channels. They consist of low-slope roofs, sloped roofs, outer walls, glazed surfaces, party walls (often not counted, the other side assumed as warm as the indoors considered) and floors, see Figure 1.52.

They are by definition three-dimensional. Quantifying the time-averaged heat flow for a 1 °C difference between in- and outdoors is done by decomposing envelopes in flat and curved parts, coupled in parallel and having as surfaces A_j and as clear wall thermal transmittances $U_{o,j}$. The contact lines between parts, the structural system applied and the details applied may add linear thermal transmittances over lengths L_k and local thermal transmittances. The result is a mean thermal transmittance, equal to:

$$U_m = \frac{\sum_{j=1}^{n}(a_j A_j U_{o,j}) + \sum_{k=1}^{m}(a_k L_k \psi_k) + \sum_{l=1}^{p}(a_l \chi_l)}{A_T} \tag{1.141}$$

with the multipliers a_i, a_k and a_l reduction factors with value 1 for parts separating the indoors from outdoors, a value <1 for parts separating the indoors from unheated adjacent spaces, from floors on grade, floors above unheated basements or crawlspaces and vertical walls contacting the soil and, if mandated, a value 0 for party walls. The a for parts contacting water channels is:

$$a = \frac{1}{1 - 0.04U_o}$$

Figure 1.52 The envelope, also called enclosure.

with U_o their clear wall thermal transmittance, calculated with a surface film coefficient h_e zero at the waterside.

Of course, how to measure related surfaces and lengths has to be decided upon. Handy is using the outside dimensions because these can be read from the architectural plans. When for simplicity reasons thermal bridging is not included, using the outside dimensions also limits the error. Bad workmanship of course can result in an average thermal transmittance larger than calculated. A formula reflecting this could be:

$$U_m = \frac{\sum_{j=1}^{n}(a_j A_j U_{o,j}/\eta_{ins,j}) + \sum_{k=1}^{m}(a_k L_k \psi_k/\eta_{ins,k}) + \sum_{l=1}^{p}(a_l \chi_l/\eta_{ins,l})}{A_T} \tag{1.142}$$

with $\eta_{ins,j}$, $\eta_{ins,k}$ and $\eta_{ins,l}$ the insulation efficiencies, 1 for perfect and <1 for lazy workmanship giving gaping joints between, air looping around, wind washing behind and indoor air washing in front of the insulation boards.

1.4.5 Heat Balances

Surface film coefficients do not exactly reflect reality. In cases, where the concept does not work, a return to solving the separate heat balances is preferred. For that, first, the surfaces or interfaces with unknown temperature and heat flow across are replaced by calculation dots, whose number fixes the totality of heat balances needed. Then, per dot, conservation of energy states that the sum of all heat flows coming from or going to the ambient or neighbour dots is zero. This way, each dot gives an equation with its temperature and the temperature of some or all of its adjacent dots as unknown and those fixed as known. Solving that system gives the

requested temperatures and heat flows or fluxes. The challenge resides in not overlooking heat flows.

1.4.6 Non-steady-state

1.4.6.1 Periodic Boundary Conditions: Flat Assemblies

Assuming that surface film resistances can be seen as the resistance of a 1 m thick air layer with thermal conductivity h_i or h_e and volumetric specific heat capacity zero turns the reference temperatures into a kind of fictitious surface temperatures, for which applies:

$$\omega_n = \sqrt{\frac{2 \text{ in } \pi\rho c\lambda}{T}} = 0 \quad \cosh(\omega_n R) = 1$$

$$\omega_n \sinh(\omega_n R) = 0 \quad \frac{\sinh(\omega_n R)}{\omega_n} = \frac{0}{0} = \lim_{n\to\infty}\left(\frac{\sinh(\omega_n R)}{\omega_n}\right) = R$$

The complex surface matrixes so become:

$$W_i = \begin{bmatrix} 1 & 1/h_i \\ 0 & 1 \end{bmatrix} \quad W_e = \begin{bmatrix} 1 & 1/h_e \\ 0 & 1 \end{bmatrix}$$

Transposing these into real ones gives:

$$W_i = \begin{bmatrix} 1 & 0 & 1/h_i & 0 \\ 0 & 1 & 0 & 1/h_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad W_e = \begin{bmatrix} 1 & 0 & 1/h_e & 0 \\ 0 & 1 & 0 & 1/h_e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.143}$$

For multi-layered envelope assemblies, the system matrix ambient to ambient so becomes:

$$W_{na} = W_i W_{n1} W_{n2} W_{n3} \cdots W_{nn} W_e$$

For a multi-layer partition, it is:

$$W_{na} = W_i W_{n1} W_{n2} W_{n3} \cdots W_{nn} W_i$$

For single-layer assemblies, the product is $W_{na} = W_i W_n W_e$ for a part belonging to the enclosure and $W_{na} = W_i W_n W_i$ for a partition or party wall.

Handling face-related transient heat exchanges this way of course is a simplification. Radiation engages all other surfaces seen, the volumetric specific heat capacity of air is not 0 but ≈1200 J/(kg·K), the air velocity plays and the interactions with other surfaces and furniture may cause some convective inertia.

1.4.6.2 Periodic Boundary Conditions: Spaces

Assumed is that the envelope and partitions enclosing a space are decomposable in flat parts coupled in parallel, while windows do not show thermal inertia. For simplicity reasons, ventilation by outside air is kept constant, air exchanges with neighbour spaces do not exist and all gains, solar and internal, are injected in the space's centre. The operative temperature θ_o in that centre is coupled to all parts by surface film coefficients h_i, combining convection and radiation (Figure 1.53).

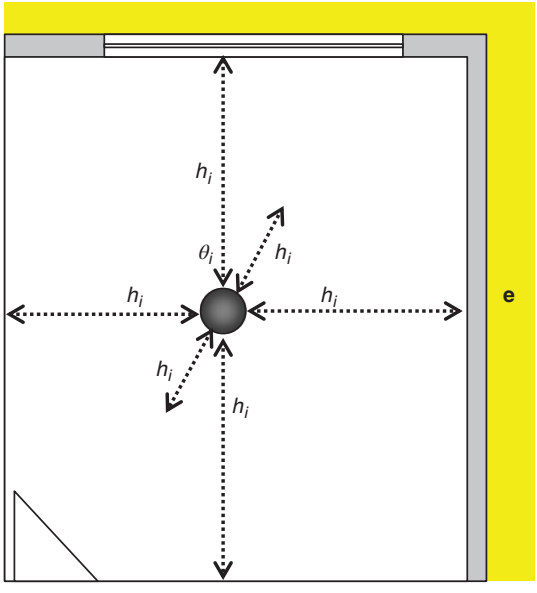


Figure 1.53 Replacing a space by its centre.

The response to a periodic heat input consists of a steady state average, a first harmonic with period T, equal to the time interval considered, for example 1 day, and higher harmonics with periods $T/2$, $T/3$. The steady state average with the operative temperature (θ_o) unknown equals:

$$\sum_{j=1}^{n}\left[a_{e,j}U_{e,j}A_{e,j}\left(\theta^{*}_{e,j}-\theta_o\right)\right]+\sum_{k=1}^{m}\left[U_{w,k}A_{w,k}\left(\theta'_{e,k}-\theta_o\right)\right]$$
$$+\sum_{l=1}^{p}\left[U_{w,l}A_{w,l}\left(\theta_e-\theta_o\right)\right]$$
$$\underbrace{+0.34nV(\theta_e-\theta_i)}_{(1)}+\underbrace{\sum_{k=1}^{m}(g_{w,k}f_{w,k}r_{w,k}E_{sun,w,k})+\overline{\Phi}_{intern}}_{(2)}=0 \qquad (1.144)$$

The suffix e stands for opaque envelope assemblies, w for windows, and i for partitions. $\theta^{*}_{e,j}$ is the sol–air temperature per opaque envelope assembly j. The temperature θ_i in the ventilation term (1) is the air temperature in the space, assumed equal to the operative temperature θ_o. Term (2) gives the solar gains through the windows. The θ_i's represent the operative temperatures in the adjacent spaces, the A's all surfaces and the U's the related clear wall thermal transmittances. V is the air volume in the space, n the ventilation rate (ach), g the solar transmittance of the windows included their shading devices, $E_{sun,w,k}$ the solar radiation touching the glass, f the ratio between glass and total window area and 'r' the shadow factor. The product $g_{w,k}f_{w,k}r_{w,k}E_{sun,w,k}$ gives the average solar and the value Φ_{intern} the average internal gain over the period considered. $\theta'_{e,k}$ finally, is the specific sol air temperature per window:

$$\theta'_{e,k} = \theta_e - \frac{120 e_L F_{w,sk}(1 - f_c)}{h_e}$$

with θ_e the air temperature outside, e_L the long wave emissivity of the glass, $F_{w,sk}$ the view factor between glass and sky, f_c the cloudiness factor and h_e the outside surface film coefficient.

The harmonics in turn write as:

$$\sum_{j=1}^{n} \Phi^n_{e,j} + \sum_{k=1}^{m} \Phi^n_{w,k} + \sum_{l=1}^{p} \Phi^n_{i,l} + \Phi^n_{vent} + \sum_{k=1}^{m} \Phi^n_{sun,w,k} + \Phi^n_{intern} = \underbrace{\left(\rho_a c_a + \frac{c_f M_f}{V}\right)}_{(3)} V_j \frac{d\theta^n_o}{dt} \tag{1.145}$$

with $\Phi^n_{e,j}$ the nth harmonic of the heat flows through the opaque envelope assemblies, $\Phi^n_{i,k}$ the nth harmonic of the heat flows through the partitions, $\Phi^n_{w,l}$ the nth harmonic of the heat flows through the windows, Φ^n_{vent} the nth harmonic of the enthalpy flow by ventilation, $\Phi^n_{sun,w,k}$ the nth harmonic of the solar gains, Φ^n_{intern} the nth harmonic of the internal gains, θ^n_o the nth harmonic of the operative temperature, c_f the specific heat capacity and M_f the weight of all furniture and furnishings in the space.

The operative temperature and heat flows, written as complex quantities, look:

Operative temperature	$\theta^n_o = \alpha^n_o \exp(2in\pi t/T)$
Transmission	$\Phi^n = \hat{\Phi}^n \exp(2in\pi t/T)$
Ventilation	$\Phi^n_{vent} = \hat{\Phi}^n_{vent} \exp(2in\pi t/T)$
Solar gains	$\Phi^n_{sun} = \hat{\Phi}^n_{sun} \exp(2in\pi t/T)$
Internal gains	$\Phi^n_{internal} = \hat{\Phi}^n_{internal} \exp(2in\pi t/T)$

In these formulas, α^n_o is the complex operative temperature, the $\hat{\Phi}^n_x$'s all complex heat flows, T the base period, n the order of the harmonic and i the imaginary unit. Rewriting the harmonic heat balance so gives:

$$\sum_{j=1}^{n} \hat{\Phi}^n_{e,j} + \sum_{k=1}^{m} \hat{\Phi}^n_{w,k} + \sum_{l=1}^{p} \hat{\Phi}^n_{i,l} + \hat{\Phi}^n_{V,j} + \sum_{k=1}^{m} \hat{\Phi}^n_{sun,w,k} + \hat{\Phi}^n_{intern} = i(\omega_n \rho_a c V)\alpha^n_o \tag{1.146}$$

with ω_n the pulsation of the nth harmonic. The value c, the equivalent specific heat capacity in the space, now is set equal to five times the specific heat capacity of air, so replacing the capacitive term (3, see equation 1.145):

$$c = c_a + c_f M_f/(\rho_a V) \approx 5c_a \approx 5000 \tag{1.147}$$

If necessary, the value of term (3) of course can be used.

Applying the concepts 'temperature damping, dynamic thermal resistance and admittance', discussed for flat assemblies, to the space, allows rewriting the separate

complex heat flows. The results are given for the first harmonic. Higher harmonics generate identical expressions, be it with the transient properties, complex temperatures and complex heat fluxes of the harmonic considered. Assuming all heat flows go from out to in, the ones through the opaque envelope assemblies write as:

$$\hat{\Phi}^{n}_{\mathrm{e},j} = \alpha'_{\mathrm{e},j} A_{\mathrm{e},j} = \left(\frac{1}{D_{\mathrm{q,e},j}} \alpha^{*}_{\mathrm{e},j} - \frac{D_{\theta,\mathrm{e},j}}{D_{\mathrm{q,e},j}} \alpha_{\mathrm{o}} \right) A_{\mathrm{e},j} = \left(\frac{1}{D_{\mathrm{q,e},j}} \alpha^{*}_{\mathrm{e},j} - A d_{\mathrm{e},j} \alpha_{\mathrm{o}} \right) A$$

For windows, the thermal transmittance remains the property intervening, giving as heat flow:

$$\hat{\Phi}^{n}_{\mathrm{w},k} = \alpha'_{\mathrm{w},k} A_{i,l} = \left[U_{\mathrm{w},k} \left(\alpha'_{\mathrm{e},k} - \alpha_{\mathrm{o}} \right) \right] A_{\mathrm{w},k}$$

The heat flows crossing the opaque partitions to adjacent spaces look:

$$\hat{\Phi}^{n}_{i,l} = \alpha'_{i,l} A_{i,l} = (\alpha_l / D_{\mathrm{q},i,l} - A d_{i,l} \alpha_{\mathrm{o}}) A_{i,l}$$

The constant ventilation rate gives as equation for the related heat flow:

$$\Phi_{\mathrm{vent}} = 0.34 \ \mathrm{nV}(\alpha_e - \alpha_o)$$

If besides the solar irradiation also the solar transmittance of a window, included its shading, varies, then the complex component of the solar gains turns into:

$$\hat{\Phi}_{\mathrm{sun,w},k} = f_{\mathrm{w},k} A_{\mathrm{w},k} \left(\alpha'_{\mathrm{sun,w},k} \right) \quad \text{with} \quad \alpha'_{\mathrm{sun,w},k} = \mathrm{Harm} \ (g_{\mathrm{w},k} f_{\mathrm{w},k} r_{\mathrm{w},k} q_{\mathrm{sun,w},k})$$

with $\alpha'_{\mathrm{sun,w},k}$ the flux touching the outside face of the shading. Harm(…) indicates that the product between brackets forms a Fourier series.

The complex components of the internal gains finally follow from a Fourier analysis:

$$\hat{\Phi}^{n}_{\mathrm{intern}} = \mathrm{Harm}(\Phi_{\mathrm{intern}})$$

Transposing all these flow equations into the balance equation and solving it for the complex operative temperature gives:

$$\alpha_{\mathrm{o}} = \frac{\left[\begin{array}{l} \sum_{j=1}^{n} \left(\frac{A_{\mathrm{e},j}}{D_{\mathrm{q,e},j}} \alpha^{*}_{\mathrm{e},j} \right) + \sum_{k=1}^{m} \left(U_{\mathrm{w},k} A_{\mathrm{w},k} \alpha'_{\mathrm{e},k} \right) + \sum_{l=1}^{p} \left(\frac{A_{i,l}}{D_{\mathrm{q},i,l}} \alpha_l \right) + 0.34 \mathrm{nV} \alpha_e \\ + \sum_{k=1}^{m} f_{\mathrm{w},k} A_{\mathrm{w},k} \mathrm{Harm}(g_{\mathrm{w},k} r_{\mathrm{w},k} q_{\mathrm{sun,w},k}) + \mathrm{Harm}(\hat{\Phi}_{\mathrm{intern}}) \end{array} \right]}{\sum_{j=11}^{n} (A_{\mathrm{e},j} A d_{\mathrm{e},j}) + \sum_{k=1}^{m} (U_{\mathrm{w},k} A_{\mathrm{w},k}) + \sum_{1=1}^{p} (A_{i,l} A d_{i,l}) + 0.34 nV + i(6000 \omega V)}$$

Using this equation in practice demands a turn to real numbers. If the sol–air and specific sol–air temperatures are presumed equal to the air temperature outdoors

($\theta'_e = \theta^*_e = \theta_e$ and $\alpha'_e = \alpha^*_e = \alpha_e$), which excludes solar radiation and under-cooling, then, for a ventilation rate zero and all adjacent spaces at the same operative temperature as the space considered, the formula simplifies to:

$$\alpha_o = \left\{ \frac{\sum_{j=1}^{n} \left(\frac{A_{e,j}}{D_{q,e,j}} \right) + \sum_{k=1}^{m} (U_{w,k} A_{w,k})}{\sum_{j=1}^{n} (A_{e,j} Ad_{e,j}) + \sum_{k=1}^{m} (U_{w,k} A_{w,k}) + \sum_{l=1}^{q} \left[A_{i,l} \left(Ad_{i,l} - \frac{1}{D_{i,l}} \right) \right] + i(6000\omega V)} \right\} \alpha_e \tag{1.148}$$

The term between large brackets contains only construction-related characteristics: surfaces, the inverses of the dynamic thermal resistance and admittance of the opaque envelope parts representing the heat storage capacity, the surface and thermal transmittance of all windows, the surface and heat storage capacity of all partitions and the heat storage capacity of the air, the furniture and the furnishings in the space, as indicated set equal to $\rho_a c \approx 6000\,\mathrm{J/(m^3 \cdot K)}$. The inverse, called the room damping for the harmonic considered, so stands for the ratio between the complex outdoor air temperature and the complex indoor operative temperature:

$$D_{\theta,\mathrm{space}} = \left\{ \frac{\sum_{j=1}^{n} (A_{e,j} Ad_{e,j}) + \sum_{k=1}^{m} (U_{w,k} A_{w,k}) + \sum_{l=1}^{q} \left[A_{i,l} \left(Ad_{i,l} - \frac{1}{D_{i,l}} \right) \right] + i(6000\omega V)}{\sum_{j=1}^{n} \left(\frac{A_{e,j}}{D_{q,e,j}} \right) + \sum_{k=1}^{m} (U_{w,k} A_{w,k})} \right\} \tag{1.149}$$

The property reflects how well the daily temperature fluctuation outdoors will be dampened after entering a space. The first harmonic usually suffices to classify spaces as showing a high or a low dampening.

1.4.6.3 Any Boundary Conditions: Thermal Bridges

For that, 'conduction, non-steady state, any boundary condition, flat assemblies' must be combined with what is discussed about the surface resistance approach under 'construction-related applications, steady state, thermal bridges'. The fictitious transposition of the surface resistance into a 1-m-thick air layer with the surface film coefficient as thermal conductivity, a capacitance 0 and heat transferred perpendicular to the end faces again applies here.

Problems and Solutions

Problem 1 Calculate the thermal transmittance of an outer wall, composed of from the inside out:

	d (cm)	λ (W/(m · K))	R (m² · K/W)
$h_i = 7.7\,\mathrm{W/(m^2 \cdot K)}$			
Plaster	1	0.3	
Inside leaf	14	0.5	
Cavity fill	8	0.04	
Unvented air cavity	4		0.17
Brick veneer	9	0.9	
$h_e = 25\,\mathrm{W/(m^2 \cdot K)}$			

Solution 1 All quantities must be in SI units. So, metres (m), not centimetres (cm):

$$\begin{aligned} U_o &= \frac{1}{1/h_i + \sum R_j + 1/h_e} \\ &= \frac{1}{1/8 + 0.01/0.3 + 0.14/0.5 + 0.08/0.04 + 0.17 + 0.09/0.9 + 1/25} \\ &= 0.36\,\mathrm{W/(m^2 \cdot K)} \end{aligned}$$

As all property values are to some extent loaded with uncertainty, limit the result to two digits.

Problem 2 Calculate the thermal transmittance of a low-slope roof, composed of from the inside out:

	d (cm)	λ (W/(m · K))
$h_i = 10\,\mathrm{W/(m^2 \cdot K)}$		
Plaster	1	0.3
Concrete floor	14	2.5
Screed	10	0.6
Vapour barrier	1	0.2
Thermal insulation	12	0.028
Membrane	1	0.2
$h_e = 25\,\mathrm{W/(m^2 \cdot K)}$		

Solution 2 $U_o = 0.21\,\mathrm{W/(m^2 \cdot K)}$

Problem 3 Calculate the clear wall thermal transmittance of a timber frame wall, composed of from the inside out (the studs concealed):

Layer	d (cm)	λ (W/(m · K))	R (m² · K/W)
$h_i = 7.7\,\mathrm{W/(m^2 \cdot K)}$			
Gypsum board	1.2	0.2	
Air space	2		0.17
Airflow retarder	0.02	0.2	
Thermal insulation	20	0.04	
Outside sheathing	2	0.14	
Unvented air cavity	2		0.17
Brick veneer	9	0.9	
$h_e = 25\,\mathrm{W/(m^2 \cdot K)}$			

Solution 3 $U_o = 0.17\,\mathrm{W/(m^2 \cdot K)}$

Problem 4 Calculate the sol–air temperature for a horizontal surface receiving 750 W/m² solar irradiation. The long wave losses to the clear sky touch 100 W/m². The outdoor air temperature is 30 °C, the outside surface film coefficient 12 W/(m²·K), the shortwave absorptivity of the outer face 0.9 and its long wave emissivity 0.8. Indoors, the surface film coefficient is 7.8 W/(m²·K). Assume steady state.

Repeat the calculation for 24 °C as daily mean outdoor air temperature, 169 W/m² as daily mean solar irradiation, 50 W/m² as daily mean long wave losses to the clear sky and the surface film coefficients just given. The numbers are representative for a south-oriented vertical wall during a hot summer day in a temperate climate.

Redo the exercise for a cold winter day with −15 °C as daily mean outdoor air temperature, 109 W/m² as daily mean solar irradiation, 50 W/m² as daily mean long wave losses if the sky were clear with 0.8 as cloudiness factor. The shortwave absorptivity and long wave emissivity of the wall's outside surface equal 0.5, respectively 0.8, the outside surface film coefficient is 16 W/(m²·K) and the one inside shows no change with the value in the first step of this exercise.

Solution 4 The sol–air temperature in the first situation is:

$$\theta_e^* = \theta_e^* + \frac{a_K E_S - e_L q_L}{h_e} = 30 + \frac{0.9 \cdot 750 - 0.8 \cdot 100}{12} = 79.6\,^\circ\mathrm{C}$$

which is high. The mean sol–air for the south-oriented wall during the summer day touches:

$$\theta_e^* = \theta_e^* + \frac{a_K E_S - e_L q_L}{h_e} = 24 + \frac{0.5 \cdot 169 - 0.8 \cdot 50}{16} = 26.8\,^\circ\mathrm{C}$$

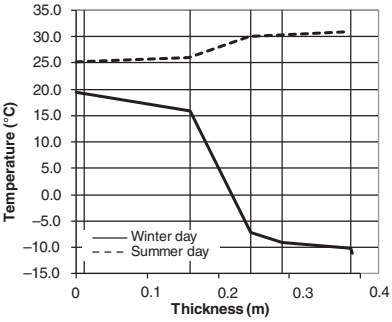
During the cold winter day, it drops to:

$$\theta_e^* = \theta_e^* + \frac{a_K E_S - e_L q_L}{h_e} = -15 + \frac{0.5 \cdot 109 - 0.8 \cdot 0.8 \cdot 50}{16} = -13.6\,^\circ\mathrm{C}$$

Problem 5 Return to Problem (1). Calculate the highest and lowest daily mean temperatures in all interfaces, knowing that the sol-air temperature has the same value as in the repeat part of Problem (4). The operative temperature indoors is 21 °C in winter and 25 °C in summer. The surface film coefficient outside is 16 W/(m²·K), the surface film resistance inside 0.13 m²·K/W. Draw the result.

Solution 5 The temperatures follow from $\theta_j = \theta_i - \left(\theta_i - \theta_e^*\right) \sum_{i=1}^{j} R/R_a$. The formula gives as table and figure:

Layer	ΣR m².K/W	Temp = cold winter day °C	Temp = warm summer day °C
	0	21.0	25.0
	0.13	19.5	25.3
1	0.16	19.1	25.4
2	0.44	15.9	26.0
3	2.44	–7.1	30.3
4	2.61	–9.0	30.6
5	2.71	–10.2	30.9
	2.78	–10.9	31.0

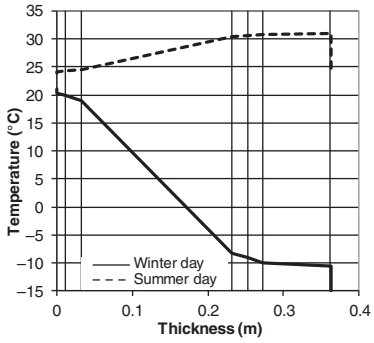


The insulation buffers most of the temperature difference, splitting the wall in an inner part with quite stable temperature and an outer part, suffering the temperature change.

Problem 6 Repeat Problem (5) for the timber frame wall of Problem (3). The outdoor sol–air and air temperature in winter and summer, the operative temperature indoors in winter and summer and the in- and outside surface film coefficients are the same as in Problem (5). Draw the result.

Solution 6

Interface	ΣR_j m².K/W	Temperature (°C) Winter	Summer
$1/h_i$	0.00	21	24
Gypsum board	0.13	20.3	24,2
Air space	0.19	20.0	24,2
AFVR	0.36	19.0	24,4
Thermal insulation	0.36	19.0	24,4
Outside sheathing	5.36	–8.3	30,4
Air cavity	5.50	–9.1	30,6
Brick veneer	5.67	–10.0	30,8
$1/h_e$	5.77	–10.6	30,9



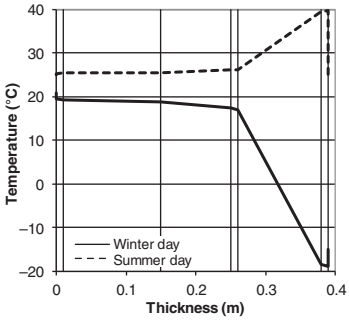
where AFVR: airflow and vapour retarder.

Problem 7 Take the low-sloped roof of Problem (2). Calculate the highest and lowest daily mean temperature in all interfaces, knowing that the daily mean sol–air temperature outdoors in summer reaches 40 °C for a daily mean outdoor air temperature of 24 °C, while in winter these values drop to −19.5 and −15 °C. The average outside surface film coefficient during windless days is 12 W/(m²·K). Inside, the

operative temperature in winter is 21 °C, in summer 25 °C. The inside surface film coefficient is 6 W/(m²·K) in summer and 10 W/(m²·K) in winter. Draw the result.

Solution 7

Interface	Winter		Summer	
	ΣR_i m².K/W	Temp. (°C)	ΣR_i m².K/W	Temp. (°C)
$1/h_i$	0	21	0.00	25.0
Render	0.17	19.6	0.10	25.3
Concrete floor	0.20	19.3	0.13	25.4
Screed	0.26	18.9	0.19	25.6
Vapor barrier	0.42	17.5	0.36	26.1
Thermal insulation	0.47	17.1	0.41	26.3
Membrane	4.76	−18.4	4.69	39.6
$1/h_e$	4.81	−18.8	4.74	39.7



Problem 8 A manufacturer introduces a new sandwich panel composed of from the inside out:

	Thickness (d) (cm)	λ (W/(m·K))	R (m²·K/W)
Aluminium	0.2	230	
VIP (vacuum insulation)	2	0.006	
Air cavity	2		0.15
Glass pane	1	Assume ∞	

The panel is used in a curtain wall. Outdoors, it's 35 °C, indoors 24 °C. Solar irradiation on the glass pane reaches 500 W/m². No long wave radiation intervenes. The outside surface film coefficient is 15 W/(m²·K), the inside 7.7 W/(m²·K). Short wave radiant properties of the glass pane are $a_S = 0.05$, $\rho_S = 0.20$, $\tau_S = 0.75$. The VIP facing the cavity behind has a shortwave absorptivity 1. Which temperature will be noted in the glass pane? How large is the heat flux entering the building through the panel?

Solution 8 The problem is solved by writing two heat balances: one for the glass pane and one for the surface of the VIP face, side of the cavity:

$$\text{Glass (temperature } \theta_1) \qquad h_e(\theta_e - \theta_1) + a_S E_S + \frac{\theta_{s2} - \theta_1}{R_{cav}} = 0$$

$$\text{VIP (temperature } \theta_{s2}) \qquad \frac{\theta_1 - \theta_{s2}}{R_{cav}} + \tau_S E_S + \frac{\theta_i - \theta_{s2}}{R_{VIP} + R_{alu} + 1/h_i} = 0$$

or:

$$\begin{cases} -(15 + 1/0.15)\theta_1 + \dfrac{\theta_{s2}}{0.15} = -0.05 \cdot 500 - 15 \cdot 35 \\ \dfrac{\theta_1}{0.15} - \theta_{s2}\left(\dfrac{1}{0.15} + \dfrac{1}{0.02/0.005 + 0.002/230 + 1/7.7}\right) \\ \quad = -0.75 \cdot 500 - \dfrac{1}{0.02/0.005 + 0.002/230 + 1/7.7} \cdot 24 \end{cases}$$

Solving this system gives $\theta_1 = 60.2\,°C$, $\theta_{s2} = 113.2\,°C$. The heat flux to the inside so is $21.6\,W/m^2$. The high temperatures show the panel acts as solar collector. Heat flux to the inside equals the one an assembly with clear wall U-value $1.96\,W/(m^2{\cdot}K)$ should transfer in the absence of solar radiation, while the clear wall U-value of the manufactured panel is only $0.23\,W/(m^2{\cdot}K)$. Which measures be taken to could lower the temperatures in the panel and the heat flux to the inside?

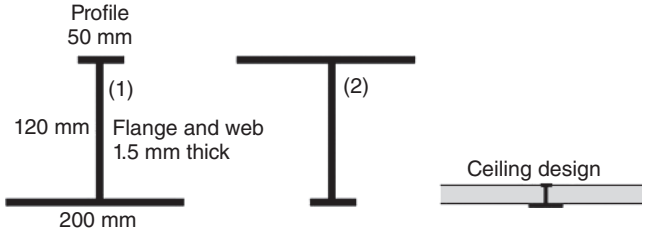
Problem 9 Solve Problem (8) for the case heat absorbing glass, $a_S = 0.3$, $r_S = 0.19$, $\tau_S = 0.51$, is used and the shortwave absorptivity and reflectivity of the VIP's face side cavity is 0.5.

Solution 9 The temperature of the glass is 52.8 °C, and the temperature at the cavity side of the VIP 70.2 °C. The heat flux to the inside equals $11.2\,W/m^2$, which corresponds to a U-value $1.02\,W/(m^2{\cdot}K)$. The real U-value remains $0.23\,W/(m^2{\cdot}K)$.

Problem 10 On the roof of a mountain chalet lays 40 cm snow ($\lambda = 0.07\,W/(m{\cdot}K)$, $a_S = 0.15$). The outdoor temperature is −15 °C, the indoor 22 °C. Solar irradiation touches $600\,W/m^2$. The outside surface film coefficient is $15\,W/(m^2{\cdot}K)$, the inside $10\,W/(m^2{\cdot}K)$. Which insulation thickness is needed to restrain the snow from melting in the contact with the membrane, if the insulation material has as apparent thermal conductivity $0.023\,W/(m{\cdot}K)$? What heat flux will cross the roof? Without insulation, the thermal resistance face to face of the roof is $0.5\,m^2{\cdot}K/W$.

Solution 10 Thickness needed is 21 cm. The heat flux across equals $2.24\,W/m^2$.

Problem 11 An intensely ventilated attic receives an insulated ceiling, composed of metal girders, mounted 60 cm centre-to-centre with 120 mm thick thermal insulation boards in between. The girder section is given in the figure below. Suppose the insulation has as thermal conductivity $0\,W/(m{\cdot}K)$, the metal $\infty\,W/(m{\cdot}K)$. The surface film coefficients are $25\,W/(m^2{\cdot}K)$ at the attic side and inside $6\,W/(m^2{\cdot}K)$. The attic temperature is −10 °C, and the temperature indoors is 20 °C. Does the heat loss differ between the profile mounted with the broader flange down or up? What is the metal temperature in both cases? Calculate the U-value of the ceiling?



Solution 11 The heat balances are:

broader flange down (1) is: $0.2{\cdot}6{\cdot}(20-\theta_x)+0.06{\cdot}25{\cdot}(-10-\theta_x)=0$,
broader flange up (2): $0.05{\cdot}6{\cdot}(20-\theta_x)+0.2{\cdot}25{\cdot}(-10-\theta_x)=0$.

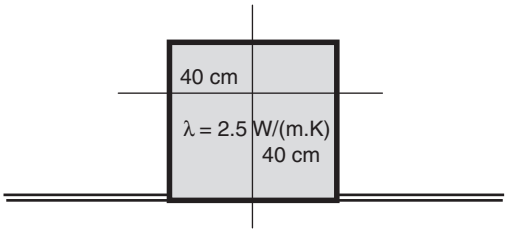
This leads to following results:

Heat loss different? Yes
Broader flange down, the metal temperature is 4.7 °C; up, it is −8.3 °C
For 1/0.6 girders per running metre broader flange down, $U = 1.02\,\mathrm{W/(m^2{\cdot}K)}$, broader flange
o up, $0.47\,\mathrm{W/(m^2{\cdot}K)}$

Problem 12 Solve Problem (11) for a metal profile with flanges of 100 mm each.

Solution 12 The temperature of the steel profiles is 4.2 °C, the *U*-value of the ceiling $0.53\,\mathrm{W/(m^2{\cdot}K)}$.

Problem 13 A reinforced concrete column with sides 0.4 m stands between two glass panels in a way the glass lines with its inner face. The glass is assumed having thickness 0 m. The temperature indoors is 21 °C, and the temperature outdoors 0 °C. The inside surface film coefficient is $8\,\mathrm{W/(m^2{\cdot}K)}$ and the outside $25\,\mathrm{W/(m^2{\cdot}K)}$. Calculate the temperature field in and the heat loss through the column?



Solution 13 Assumed first is that the column reacts as a flat wall. The *U*-value then is:

$$\frac{1}{1/25 + 0.4/2.5 + 1/8} = 3.1\,\mathrm{W/(m^2{\cdot}K)}.$$

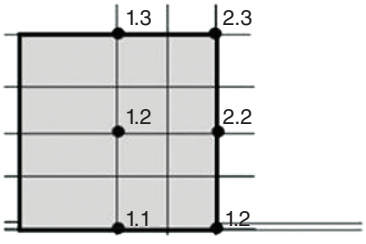
The temperature at the inside surface so equals $21 - 3.1{\cdot}(21 - 0)/8 = 12.9$ °C, giving a temperature factor 0.65. The heat loss equals $3.1{\cdot}0.4{\cdot}21 = 25.8\,\mathrm{W/m}$. Or, the temperature factor and thermal transmittance are close to double glass. In the column's centre, the temperature is 7.8 °C.

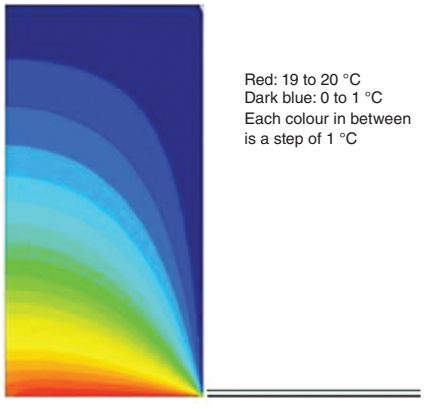
A first upgrade consists of applying a very simple CVM grid with one centre point, the column's centre (point 1, 2). Its heat balance is:

$$\frac{0.2(21 - \theta_x)}{1/8 + 0.2/2.5} + 3\frac{0.2(0 - \theta_x)}{1/25 + 0.2/2.5} = 0$$

giving as central temperature 3.4 °C and as inside surface temperature 10.3 °C, convening with a temperature factor 0.49, a value 24.6% lower than what the flat wall assumption gave. Heat loss now is 34.3 W/m, so 33.7% higher than with the flat wall assumption.

In a second upgrade, the grid over half the column gets up to a way 6 centre points, of which 5 on the perimeter.





The heat balances now become:

$$\text{Point } 1.1 \quad 8\cdot0.1\cdot(21-\theta_{1.1})+\frac{2.5\cdot0.1}{0.2}(\theta_{1.2}-\theta_{1.1})+\frac{2.5\cdot0.1}{0.2}(\theta_{2.1}-\theta_{1.1})=0$$

$$\text{Point } 2.1 \quad 8\cdot0.1\cdot(21-\theta_{2.1})+\frac{2.5\cdot0.1}{0.2}(\theta_{1.1}-\theta_{2.1})+\frac{2.5\cdot0.1}{0.2}(\theta_{2.2}-\theta_{2.1})+25\cdot0.1\cdot(0-\theta_{2.1})=0$$

$$\text{Point } 1.2 \quad \frac{2.5\cdot0.1}{0.2}(\theta_{1.1}-\theta_{1.2})+\frac{2.5\cdot0.1}{0.2}(\theta_{1.3}-\theta_{1.2})+\frac{2.5\cdot0.2}{0.2}(\theta_{2.2}-\theta_{1.2})=0$$

$$\text{Point } 2.2 \quad \frac{2.5\cdot0.1}{0.2}(\theta_{2.1}-\theta_{2.2})+\frac{2.5\cdot0.2}{0.2}(\theta_{1.2}-\theta_{2.2})+\frac{2.5\cdot0.1}{0.2}(\theta_{2.3}-\theta_{2.2})+25\cdot0.2\cdot(0-\theta_{2.2})=0$$

$$\text{Point } 1.3 \quad \frac{2.5\cdot0.1}{0.2}(\theta_{1.2}-\theta_{1.3})+25\cdot0.1\cdot(0-\theta_{1.3})+\frac{2.5\cdot0.1}{0.2}(\theta_{2.3}-\theta_{1.3})=0$$

$$\text{Point } 2.3 \quad 2\cdot25\cdot0.1\cdot(0-\theta_{2.3})+\frac{2.5\cdot0.1}{0.2}(\theta_{2.2}-\theta_{2.3})+\frac{2.5\cdot0.1}{0.2}(\theta_{1.3}-\theta_{2.3})=0$$

Solving this system gives as temperatures in the column:

0.4 0.9 0.4

1.4 2.9 1.4

4.9 8.1 4.9

The lowest temperature factor inside now is at the corners, 0.25, as bad as single glass. The heat loss turns into:

$$\Phi = 2\cdot0.1\cdot8\cdot(21-4.9)+0.2\cdot8\cdot(21-8.1)=46.4\,\mathrm{W/m}$$

which is 80% higher than with the flat wall assumption.

Last upgrade is calculated using software for two-dimensional heat transport. The figure above at the right shows the isotherms found, nearly the correct answer in terms of temperatures.

Problem 14 Aerated concrete is chosen as material for the outer walls. The manufacturer praises its good transient thermal response, ensuring a much higher dynamic thermal resistance than the steady state value. Is this true? The material properties are:

Situation	ρ (kg/m^3)	λ (W/(m·K))	c (J/(kg·K))
Just applied (humid)	450	0.30	2700
After some years (air-dry)	450	0.13	1120

The wall thickness is either 10, 20 or 30 cm. The outside surface film coefficient touches 25 W/(m^2·K), the inside 8 W/(m^2·K).

Solution 14 The truth relies in the wall's harmonic properties. A high dynamic thermal resistance may confirm the manufacturer's claim but a low admittance says this does not suffice to stabilize the indoor conditions in case of important solar and internal gains. To show, temperature damping, the dynamic thermal resistance and the admittance are calculated for a humid 10 cm thick wall.

$$\text{Thermal diffusivity: } a = \frac{\lambda}{\rho c} = 2.469 \cdot 10^{-7}\,\text{m}^2/\text{s} \ .$$

$$X_n\text{-value:} \quad X_n = d\sqrt{\frac{n\pi}{aT}} = 0.1\sqrt{\frac{3.1418}{2.469 \cdot 10^{-7} \cdot 3600 \cdot 24}} = 1.2135$$

$$\begin{array}{llll} G_{n1} & 0.640 & G_{n4} & 0.486 \\ G_{n2} & 1.437 & G_{n5} & -1.431 \\ G_{n3} & 0.928 & G_{n6} & 2.733 \end{array}$$

Functions $G_{n1}(X_n)$ to $G_{n6}(X_n)$:
Layer matrices

Inside surface (1)

$$\begin{vmatrix} 1 & 0 & 0.125 & 0 \\ 0 & 1 & 0 & 0.125 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Layer (2)

$$\begin{vmatrix} 0.640429 & 1.437199 & 0.309307 & 0.161937 \\ -1.437199 & 0.640429 & -0.161937 & 0.309307 \\ -4.29249 & 8.198851 & 0.640429 & 1.437199 \\ -8.198851 & -4.29249 & -1.437199 & 0.640429 \end{vmatrix}$$

Outside surface (3)

$$\begin{vmatrix} 1 & 0 & 0.04 & 0 \\ 0 & 1 & 0 & 0.04 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Matrix multiplication:

$$\text{Layer (2)} \cdot \text{layer (2)} = \begin{vmatrix} 0.640429 & 1.437199 & 0.38936 & 0.341587 \\ -1.437199 & 0.640429 & -0.341587 & 0.38936 \\ -4.29249 & 8.198851 & 0.103868 & 2.462055 \\ -8.198851 & -4.29249 & -2.462055 & 0.103868 \end{vmatrix} \text{(matrix 4)}$$

$$\text{Layer (3)} \cdot \text{(matrix 4)} = \begin{vmatrix} 0.46873 & 1.765153 & 0.3993515 & 0.440069 \\ -1.765153 & 0.46873 & -0.440069 & 0.3993515 \\ -4.29249 & 8.198851 & 0.103868 & 2.462055 \\ -8.198851 & -4.29249 & -2.462055 & 0.103868 \end{vmatrix}$$

This gives as harmonic properties:

$$\text{Temperature damping} \quad |D_\theta| = \sqrt{0.46873^2 + 1.765133^2} = 1.83$$

$$\varphi_\theta = a \tan\left(\frac{1.765133}{0.46873}\right)\frac{12}{\pi} = 5 \text{ h}$$

$$\text{Dynamic thermal resistance} \quad |D_q| = \sqrt{0.3993515^2 + 0.38936^2}$$

$$= 0.59 \text{ m}^2\cdot\text{K/W}$$

$$\varphi_q = a \tan\left(\frac{0.38936}{0.3993515}\right)\frac{12}{\pi} = 3.2 \text{ h}$$

$$\text{Admittance} \quad |\text{Ad}| = \frac{|D_\theta|}{|D_\text{q}|} = 3.09 \text{ W/(m}^2\cdot\text{K)}$$

$$\varphi_\text{Ad} = \varphi_\theta - \varphi_\text{q} = 1.8 \text{ h}$$

It's up to the reader to calculate on spreadsheet the three for a 10 cm air-dry aerated concrete outer wall and for a 20 and 30 cm thick humid and air-dry aerated concrete outer wall. The results should be:

	Humid			Air-dry		
	Case 1 $d = 10$ cm $\lambda = 0.3$ W/(m·K)	Case 2 $d = 20$ cm $\lambda = 0.3$ W/(m·K)	Case 3 $d = 30$ cm $\lambda = 0.3$ W/(m·K)	Case 4 $d = 10$ cm $\lambda = 0.13$ W/(m·K)	Case 5 $d = 20$ cm $\lambda = 0.13$ W/(m·K)	Case 6 $d = 30$ cm $\lambda = 0.13$ W/(m·K)
D_θ	1.83	6.56	23.1	1.63	5.72	19.1
ϕ_θ (h)	5 h 00′	9 h 48′	14 h 36′	4 h 33′	9 h 19′	13 h 57′
D_q (m²·K/W)	0.59	1.93	6.56	1.02	3.16	10.4
ϕ_q (h)	3 h 18′	7 h 54′	12 h 42′	2 h 24′	6 h 53′	11 h 28′
Ad (W/(m²·K)	3.09	3.39	3.53	1.60	1.81	1.84
ϕ_{Ad} (h)	1 h 48′	1 h 52′	1 h 57′	2 h 09′	2 h 26′	2 h 29′

Clearly, aerated concrete is not the wonder in terms of thermal inertia manufacturers promise. To get a sufficient temperature damping air-dry ($D_\theta > 15$), a thickness beyond 20 cm is needed. The same holds for the dynamic thermal resistance if a value ≥ 4 m².K/W is the target. Still, the admittance remains low, surely once the aerated concrete is air-dried. The material so does not function the way claimed. On the contrary, it does not help a lot in damping the effects of the daily swings in outdoor temperature and solar radiation on the temperature indoors.

Problem 15 A living room has as surface 4×6.5 m² and a ceiling height of 2.5 m. Two of the outer walls, one 4×2.5 m², the other 6.5×2.5 m² large, are completely glazed with gas-filled, low-e double glazing, U-value 1.3 W/(m².K) for $h_{i-} = 7.7$ W/(m².K) and $h_{e-} = 25$ W/(m².K). The 2 partitions and the ceiling have a thermal resistance 0.505 m².K/W between their face in the living room and the ambient in the neighbour room. The floor heating consists of a network of pipes covered by a screed having a thermal resistance 0.1 m².K/W. Walls, floor and ceiling have a long wave emissivity 0.9, the glass 0.92. The ventilation rate touches 1 ach, while the surface film coefficient for convection is 3.5 W/(m².K). Calculate the glass, wall and ceiling temperatures, knowing that the in- and outside air temperatures are 21 and −8 °C.

Solution 15 The room is a six grey surfaces combination: two windows, one with surface A_1, temperature T_{s1} and one with surface A_2, temperature T_{s2}, two partitions, one with surface A_3, temperature T_{s3} and one with surface A_4, temperature T_{s4}, a ceiling with surface A_5, temperature T_{s5}, the floor with surface A_6, temperature T_{s6}. The water in the floor heating has a temperature T_{fl}. Seven heat balances so are needed, one convective for the room as a whole and one per wall.

$$\text{Room:} \quad Q_v + \sum_{j=1}^{6} h_c A_j (21 - \theta_{sj}) = 0,$$

or, with $Q_v = \rho_a c_a V(\theta_e - \theta_i)$, $\theta_i = 21\,^\circ\text{C}$, $\theta_e = -8\,^\circ\text{C}$, $V = 65\,\text{m}^3$, $A_1 = A_3 = 10\,\text{m}^2$, $A_2 = A_4 = 16.25\,\text{m}^2$, $A_5 = A_6 = 26\,\text{m}^2$, $c_a = 1008\,\text{J/(kg·K)}$, ρ_a and $h_c = 3.5\,\text{W/(m}^2\text{·K)}$:

$$-633.36 + 35T_{s1} + 56.875T_{s2} + 35T_{s3} + 56.875T_{s4} + 91T_{s5} + 91T_{s6} - 7680.75 = 0$$

Wall surface-related heat balances: the radiant heat flux writes as: $q_R = e_L (M' - M_b)/\rho_L$. Linearization of the black body emittance M_b in the temperature interval 10–25 °C gives:

$$M_b = 307.75 + 5.57\ \theta_s, \quad r^2 = 0.999$$

The radiosity $M'_j = \frac{M_{bj}}{e_j} - \frac{\rho_j}{e_j}\sum_{i=2}^{6} F_{ji} M'_j$ with F_{ji} view factor between each face and the other 5:

	Surface 1	Surface 2	Surface 3	Surface 4	Surface 5	Surface 6
Surface 1	—	0.187	0.070	0.187	0.278	0.278
Surface 2	0.115	—	0.115	0.210	0.280	0.280
Surface 3	0.070	0.187	—	0.187	0.278	0.278
Surface 4	0.115	0.210	0.115	—	0.280	0.280
Surface 5	0.107	0.175	0.107	0.175	—	0.436
Surface 6	0.107	0.175	0.107	0.175	0.436	—

The black body emittance of the surfaces so becomes:

$$\text{s1}\ \ M_{b1} = \frac{1}{0.92}M'_{s1} - \frac{0.08}{0.92} \cdot \left(0.187M'_{s2} + 0.07M'_{s3} + 0.187M'_{s4} + 0.278M'_{s5} + 0.278M'_{s6}\right)$$

$$\text{s2}\ \ M_{b2} = \frac{1}{0.92}M'_{s2} - \frac{0.08}{0.92} \cdot \left(0.115M'_{s1} + 0.115M'_{s3} + 0.21M'_{s4} + 0.28M'_{s5} + 0.28M'_{s6}\right)$$

$$\text{s3}\ \ M_{b3} = \frac{1}{0.9}M'_{s3} - \frac{0.1}{0.9} \cdot \left(0.07M'_{s1} + 0.187M'_{s2} + 0.187M'_{s4} + 0.278M'_{s5} + 0.278M'_{s6}\right)$$

$$\text{s4}\ \ M_{b4} = \frac{1}{0.9}M'_{s4} - \frac{0.1}{0.9} \cdot \left(0.115M'_{s1} + 0.210M'_{s2} + 0.115M'_{s3} + 0.28M'_{s5} + 0.28M'_{s6}\right)$$

$$\text{s5}\ \ M_{b5} = \frac{1}{0.9}M'_{s5} - \frac{0.1}{0.9} \cdot \left(0.107M'_{s1} + 0.175M'_{s2} + 0.107M'_{s3} + 0.175M'_{s4} + 0.436M'_{s6}\right)$$

$$\text{s6}\ \ M_{b6} = \frac{1}{0.9}M'_{s6} - \frac{0.1}{0.9} \cdot \left(0.107M'_{s1} + 0.175M'_{s2} + 0.107M'_{s4} + 0.175M'_{s4} + 0.436M'_{s5}\right)$$

Inverting the matrix of this system of six equations gives the radiosities of the six surfaces as function of their black surface emittances:

$$\text{Matrix}: \begin{vmatrix} 1.0870 & -0.0163 & -0.0061 & -0.0163 & -0.0242 & -0.0242 \\ -0.0100 & 1.0870 & -0.0100 & -0.0183 & -0.0243 & -0.0243 \\ -0.0078 & -0.0208 & 1.1111 & -0.0208 & -0.0309 & -0.0309 \\ -0.0128 & -0.0233 & -0.0128 & 1.1111 & \cdot 0.0311 & -0.0311 \\ -0.0119 & -0.0194 & -0.0119 & -0.0194 & 1.1111 & -0.0485 \\ -0.0119 & -0.0194 & -0.0119 & -0.0194 & -0.0485 & 1.1111 \end{vmatrix}$$

$$\text{Inverted} \begin{vmatrix} H'_1 \\ H'_2 \\ H'_3 \\ H'_4 \\ H'_5 \\ H'_6 \end{vmatrix} = \begin{vmatrix} 0.9208 & 0.0150 & 0.0058 & 0.0146 & 0.0219 & 0.0219 \\ 0.0092 & 0.9214 & 0.0090 & 0.0162 & 0.0221 & 0.0221 \\ 0.0074 & 0.0187 & 0.9010 & 0.0182 & 0.0273 & 0.0273 \\ 0.0115 & 0.0207 & 0.0112 & 0.9017 & 0.0275 & 0.0275 \\ 0.0108 & 0.0176 & 0.0105 & 0.0172 & 0.9032 & 0.0408 \\ 0.0108 & 0.0176 & 0.0105 & 0.0172 & 0.0408 & 0.9032 \end{vmatrix}$$

$$\cdot \begin{vmatrix} 307.75 + 5.57\theta_{s1} \\ 307.75 + 5.57\theta_{s2} \\ 307.75 + 5.57\theta_{s3} \\ 307.75 + 5.57\theta_{s4} \\ 307.75 + 5.57\theta_{s5} \\ 307.75 + 5.57\theta_{s6} \end{vmatrix}$$

Introducing this result in the radiant heat flux equation allows eliminating the constant 307.75. The radiant, convective and conductive heat balance per surface now is $q_R + q_C + q_{cond} = 0$, or:

$$\begin{aligned} -10.1367\theta_{s1} + 0.9599\theta_{s2} + 0.3726\theta_{s3} + 0.9352\theta_{s4} \\ + 1.4023\theta_{s5} + 1.4023\theta_{s6} + 0\theta_{fl} = 60.985 \\ 0.5907\theta_{s1} - 10.0972\theta_{s2} + 0.5771\theta_{s3} + 1.0391\theta_{s4} \\ + 1.4129\theta_{s5} + 1.4129\theta_{s6} + 0\theta_{fl} = 60.985 \\ 0.3726\theta_{s1} + 0.9378\theta_{s2} - 10.444\theta_{s3} + 0.9136\theta_{s4} \\ + 1.3699\theta_{s5} + 1.3699\theta_{s6} + 0\theta_{fl} = 115.084 \\ 0.5755\theta_{s1} + 1.0391\theta_{s2} + 0.5622\theta_{s3} - 10.410\theta_{s4} \\ + 1.3765\theta_{s5} + 1.3765\theta_{s6} + 0\theta_{fl} = 115.084 \\ 0.5393\theta_{s1} + 0.8831\theta_{s2} + 0.5269\theta_{s3} + 0.8603\theta_{s4} \\ - 10.335\theta_{s5} + 2.0454\theta_{s6} + 0\theta_{fl} = 115.084 \\ 0.5393\theta_{s1} + 0.8831\theta_{s2} + 0.5269\theta_{s3} + 0.8603\theta_{s4} \\ + 2.0454\theta_{s5} - 18.355\theta_{s6} + 10\theta_{fl} = -73.5 \end{aligned}$$

The diagonal terms in these equations come from:

$$\theta_{s1}, \theta_{s2} - \left[\frac{1}{(1/1.3 - 0.13)} + 3.5 + 0.9208\left(\frac{0.92}{0.08}\right)\right]$$

$$\theta_{s3}, \theta_{s4}, \theta_{s5} - \left[\frac{1}{0.505} + 3.5 + (0.901,\ \ 0.9017,\ \ 0.9032)\left(\frac{0.9}{0.1}\right)\right]$$

$$\theta_{s6} - \left[\frac{1}{0.1} + 3.5 + 0.9032\left(\frac{0.9}{0.1}\right)\right]$$

Solving this system of six surface and one room heat balances gives as temperatures:

Window $4.0 \times 2.5\,m^2$	$\theta_{s1} = 17.6\,°C$	Wall $6.5 \times 2.5\,m^2$	$\theta_{s4} = 21.8\,°C$
Window $6.5 \times 2.5\,m^2$	$\theta_{s2} = 17.8\,°C$	Ceiling	$\theta_{s5} = 22.3\,°C$
Wall $4.0 \times 2.5\,m^2$	$\theta_{s3} = 21.9\,°C$	Floor	$\theta_{s6} = 29.2\,°C$

The water in the floor heating system has a temperature of 36.1 °C

Problem 16 Repeat (15) assuming the two envelope walls consist of double glazing, $U = 2.9\,W/(m^2.K)$ for $h_i = 7.7\,W/(m^2{\cdot}K)$ and $h_e = 25\,W/(m^2{\cdot}K)$, while all other data remain the same.

Solution 16 With normal double glazing, the temperatures become:

Window $4.0 \times 2.5\,m^2$	$\theta_{s1} = 11.8\,°C$
Window $6.5 \times 2.5\,m^2$	$\theta_{s2} = 12.1\,°C$
Wall $4.0 \times 2.5\,m^2$	$\theta_{s3} = 21.9\,°C$
Wall $6.5 \times 2.5\,m^2$	$\theta_{s4} = 21.7\,°C$
Ceiling	$\theta_{s5} = 22.6\,°C$
Floor	$\theta_{s6} = 34.7\,°C$
Floor heating	$\theta_{fl} = 47.0\,°C$

The floor is much warmer now than acceptable for feet comfort (28 °C), or, heat loss is too high to only install floor heating. The room also needs a radiator or a convector.

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