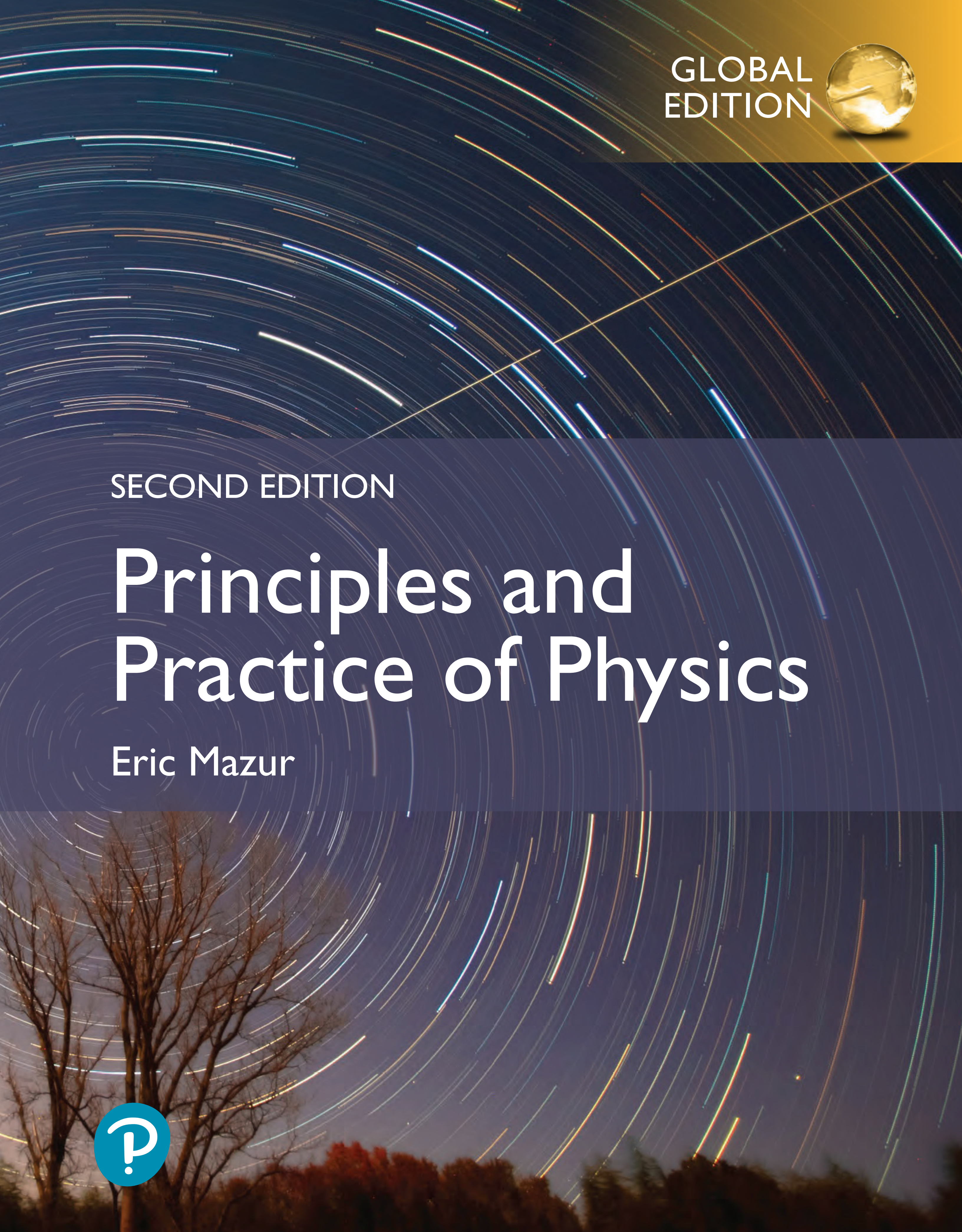
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PRINCIPLES & PRACTICE OF PHYSICS

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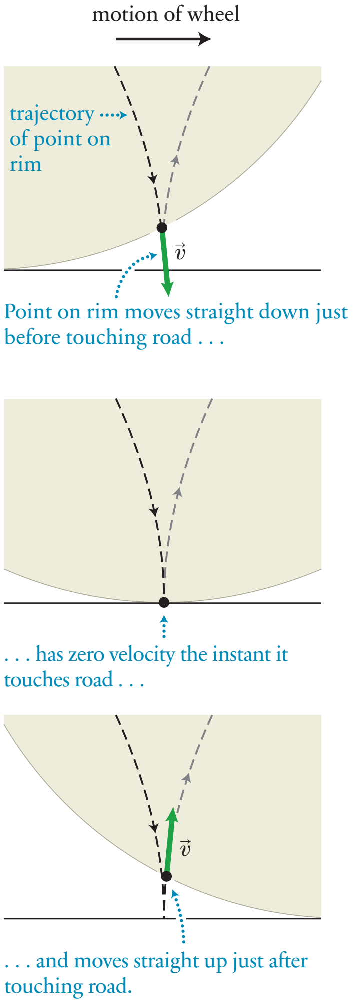
Figure 12.38 The point of contact of a wheel rolling without slipping has zero instantaneous velocity.



moves with a continuously changing velocity along a trajectory resembling that of the handle of the wrench in Figure 12.9. Note in particular that whenever a point on the rim is in contact with the surface over which the wheel is rolling, that point has zero instantaneous velocity (on the rim $r = R$ and $v_t = r\omega_\vartheta$ points in the direction opposite the direction of $v_{cm} = R\omega$). This zero instantaneous velocity of the rim point in contact with the surface is a direct consequence of the requirement that the object rolls without slipping: The relative velocity of the two surfaces in contact must be zero for there to be no slipping. **Figure 12.38** shows that points on the rim move in a direction perpendicular to the surface just before and just after reaching the bottom.

Let us next turn to the dynamics of rolling motion. Suppose a round object of inertia m and radius R is released from rest on a ramp as shown in **Figure 12.39*a***. Why does the object roll down instead of sliding down as would a block of wood? The reason is that the force of static friction, which forces the instantaneous point of contact to be motionless, causes a torque about the center of the object. To understand the motion of the object, we must find the vector sum of the forces exerted on it and the sum of the torques caused by these forces. Figure 12.39*b* shows a free-body diagram for the object, which is subject to two forces: the force of gravity and a contact force exerted by the ramp, which has a normal component $\vec{F}^n_{\text{ro}}$ in the y direction and a tangential component $\vec{F}^s_{\text{ro}}$ in the x direction due to static friction. As we have done with other problems involving ramps, we chose the x axis to be along the ramp. For the object to accelerate down the ramp, the vector sum of the forces must point in the positive x direction, so $\sum F_y = F^G_{\text{Eo}\,y} + F^n_{\text{ro}} = 0$ and $\sum F_x = F^G_{\text{Eo}\,x} - F^s_{\text{ro}} > 0$. The vector sum of the forces and the center-of-mass acceleration are related by

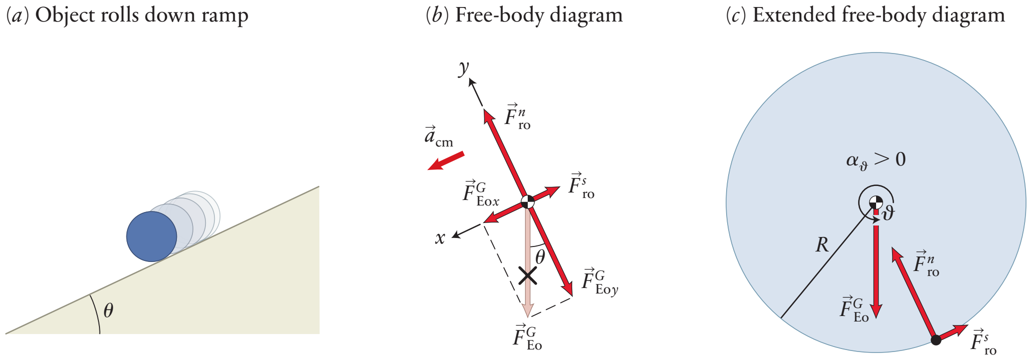
$$\sum F_x = F^G_{\text{Eo}\,x} - F^s_{\text{ro}} = mg\sin\theta - F^s_{\text{ro}} = ma_{\text{cm}\,x}. \tag{12.21}$$

This equation contains two unknowns: the magnitude of the force of static friction F^s_{ro} and $a_{\text{cm}\,x}$. All we know about F^s_{ro} is that it must be less than or equal to its maximum value $(F^s_{\text{ro}})_{\text{max}}$, but this fact doesn't help us solve Eq. 12.21. Let us therefore turn to the extended free-body diagram shown in Figure 12.39*c*, which shows the object with the forces exerted on it at their points of application. Because the object rotates about its center of mass, we determine the sum of the torques about the center of mass. Neither $\vec{F}^G_{\text{Eo}}$ nor $\vec{F}^n_{\text{ro}}$ causes a torque because their lines of action go through the center of mass, and so the lever arm distances for these forces are zero. Only $\vec{F}^s_{\text{ro}}$ causes a torque; its lever arm distance is the radius R of the object. Equation 12.10 then yields

$$\sum \tau_{\text{ext}\,\vartheta} = +F^s_{\text{ro}}R = I\alpha_\vartheta, \tag{12.22}$$

where I is the object's rotational inertia and α_ϑ is its rotational acceleration. Although this yields another equation, we have now added a third unknown: α_ϑ.

Figure 12.39 A round object rolls down a ramp.



Because the object is rolling without slipping, its rotational and translational motions are coupled, and so its rotational and center-of-mass accelerations must be related. Differentiating both sides of Eq. 12.19, $v_{\text{cm}\,x} = R\omega_\vartheta$, with respect to time yields

$$a_{\text{cm}\,x} = R\alpha_\vartheta. \tag{12.23}$$

Substituting α_ϑ from Eq. 12.23 into Eq. 12.22 and solving for F^s_{ro}, we find

$$F^s_{\text{ro}} = \frac{I}{R^2}a_{\text{cm}\,x}. \tag{12.24}$$

Substituting this result for F^s_{ro} into Eq. 12.21 and solving for $a_{\text{cm}\,x}$, we obtain

$$a_{\text{cm}\,x} = +\frac{g\sin\theta}{1+\dfrac{I}{mR^2}} = +\frac{g\sin\theta}{1+c}, \tag{12.25}$$

where $c \equiv I/mR^2$ is the *shape factor* of the object ($0 < c \leq 1$; see Table 11.3). The numerator in Eq. 12.25 is the acceleration of an object sliding down a ramp with negligible friction (see Eq. 3.20). The denominator shows by what factor this acceleration is reduced for rolling motion. Because c is independent of m and R, Eq. 12.25 shows that the acceleration of the object does not depend on the values of m and R. Only the shape of the object matters because it affects the shape factor c. For all objects of a given shape, the time interval to roll down a ramp is thus the same. A large cylinder takes the same time interval as a small one, but a thin cylindrical shell takes longer than a solid cylinder because the shape factor of the shell is greater.

We can now also find an expression for the magnitude of the force of static friction $\vec{F}^s_{\text{ro}}$. Substituting $a_{\text{cm}\,x}$ from Eq. 12.25 into Eq. 12.24 and eliminating the rotational inertia using $c \equiv I/mR^2$, which gives $I = cmR^2$, we obtain

$$F^s_{\text{ro}} = \frac{I}{R^2}a_{\text{cm}\,x} = \frac{cmR^2}{R^2}\frac{g\sin\theta}{1+c} = \frac{mg\sin\theta}{c^{-1}+1}. \tag{12.26}$$

Note that the force of static friction plays a dual role. It decreases the center-of-mass speed and acceleration of the rolling object by reducing the magnitude of the vector sum of the forces exerted on it: $\sum F_x = F^G_{\text{Eo}\,x} - F^s_{\text{ro}}$. It also causes the torque that gives the object a rotational acceleration. In the absence of static friction, there would be no torque and objects would never roll—they would only slide.

Wheels certainly make it easier to push objects, but as Eq. 12.26 shows, they do not eliminate friction. In fact, the force of static friction to which wheels are subject slows them down, and as we have seen, the force of static friction can be *greater* than the force of kinetic friction. The next examples help clarify this paradox.

12.10 (*a*) A cylindrical shell and a solid cylinder, made of the same material and having the same inertia m and radius R, roll down a ramp. Is the force of static friction exerted on the shell greater than, smaller than, or equal to that on the solid cylinder? Explain. (*b*) For $\mu_s = 1$, what is the maximum angle of incline θ on which these objects can roll without slipping?

EXAMPLE 12.7 Bicycle friction

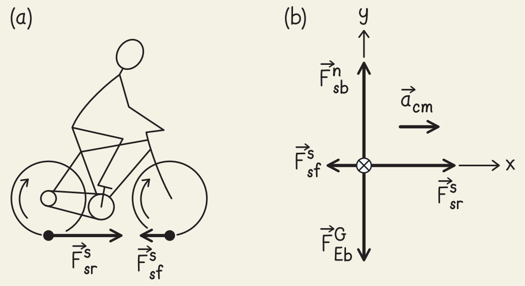
As you accelerate from rest on a bicycle, how does the magnitude of the force of friction exerted by the road surface on the rear wheel compare with the magnitude of the force of friction exerted by the road surface on the front wheel? Ignore air resistance, assume both wheels have the same inertia m_w and the same radius R, and let m_{comb} be the combined inertia of you and your bicycle (including both wheels).

❶ **GETTING STARTED** I begin by making a sketch of my bicycle (**Figure 12.40*a***). As I push on the pedals, the chain causes a torque on the rear wheel that makes the wheel rotate so that it exerts a rearward force on the road. Consequently the road surface exerts on the rear wheel a contact force in the opposite direction, and the tangential component of this force—the force of static friction—is what makes the bicycle go forward. This force, which I label $\vec{F}^s_{sr}$, is one of the forces I need for my comparison. (Note the subscripts here: r stands for "rear," not "road"; this is the frictional force exerted by the (road) surface on the rear wheel.)

The front wheel, because it is not connected to the chain, is merely pushed along with the rest of the bicycle. To make this wheel rotate in the right direction, the road surface must exert on it a rearward-pointing force of static friction $\vec{F}^s_{sf}$ (subscript f for "front" here). The torque caused by this frictional force is what makes the front wheel rotate, and this is the second force in my comparison.

Armed with this information, I make a free-body diagram for the bicycle (Figure 12.40*b*), showing the two frictional forces, the upward component of the contact force $\vec{F}^n_{sb}$ exerted by the road surface, and the downward force of gravity $\vec{F}^G_{Eb}$. Because the bicycle accelerates forward, I draw the vector arrow for $\vec{F}^s_{sr}$ longer than the vector arrow for $\vec{F}^s_{sf}$. Finally, I show the direction in which the bicycle's center of mass accelerates and choose a set of axes, letting the x axis point in the direction of motion. I also choose clockwise as the direction of increasing ϑ because that is the direction in which the wheels rotate.

Figure 12.40



❷ **DEVISE PLAN** The equation of motion in the form given by Eq. 8.45, $\vec{a}_{cm} = \sum\vec{F}_{ext}/m$, relates the vector sum of the forces shown in my free-body diagram to the acceleration $\vec{a}_{cm}$ of the bicycle's center of mass. Thus this equation gives me an expression that contains the two forces I'm after and seems a good place to begin. This expression contains the translational acceleration, however, whereas the variables in this problem are rotational, so I will probably need Eq. 12.23 relating α_ϑ and $a_{cm\,x}$. Focusing on the front wheel first, I can use the rotational equation of motion, Eq. 12.5, $\tau_\vartheta = I\alpha_\vartheta$, to relate the torque caused by $\vec{F}^s_{sf}$ to α_ϑ for this wheel and then use Table 11.3 to express everything in terms of the given variables m_w and R. Once I have my expression for the magnitude F^s_{sf}, I can work with Eq. 8.45 to obtain an expression for F^s_{sr} in terms of F^s_{sf} and make my comparison.

❸ **EXECUTE PLAN** The vector sum of the forces in the x direction is, from Eq. 8.45,

$$\sum F_x = F^s_{sr} - F^s_{sf} = m_{comb}\,a_{cm\,x}, \tag{1}$$

so I have an expression that contains both of the forces I'm after.

According to Eq. 12.23, the rotational acceleration of the rear wheel is $\alpha_\vartheta = a_{cm\,x}/R$, and the rotational acceleration of the front wheel must be the same as that of the rear wheel. To give the front wheel rotational acceleration α_ϑ, the force $\vec{F}^s_{sf}$ must cause a torque $\tau_\vartheta = I\alpha_\vartheta$ on the wheel, where I is the wheel's rotational inertia. If I treat the wheel as a thin hoop of inertia m_w, I have, from Table 11.3, $I = m_w R^2$, which means the torque on the front wheel is $\tau_\vartheta = m_w R^2\alpha_\vartheta$. The lever arm distance for this torque is equal to the radius R of the wheel, and so for this wheel $\tau_\vartheta = F^s_{sf}R = m_w R^2\alpha_\vartheta$. I know that for both wheels $\alpha_\vartheta = a_{cm\,x}/R$, however, so I can write

$$F^s_{sf}R = (m_w R^2)\left(\frac{a_{cm\,x}}{R}\right).$$

Dividing both sides by R yields one of the two forces I must compare:

$$F^s_{sf} = m_w\,a_{cm\,x}. \tag{2}$$

This is the magnitude of the force of static friction required to make the front wheel roll when the bicycle has acceleration $a_{cm\,x}$. The smaller the inertia of the wheel, the smaller this force magnitude.

Turning to the other force I need for my comparison—the force exerted on the rear wheel—I begin by eliminating $a_{cm\,x}$ from Eq. 1 using Eq. 2 to get

$$F^s_{sr} - F^s_{sf} = \frac{m_{comb}}{m_w}F^s_{sf}$$

or
$$F^s_{sr} = F^s_{sf}\left(1 + \frac{m_{comb}}{m_w}\right) = F^s_{sf}\left(\frac{m_{comb} + m_w}{m_w}\right).$$

The magnitude of the force of static friction exerted on the rear wheel is greater than that exerted on the front wheel by the factor $(m_{comb} + m_w)/m_w$. ✔

❹ **EVALUATE RESULT** Because $m_{comb} \gg m_w$, the magnitude of the force of static friction exerted on the front wheel is much smaller than that on the rear wheel, as I expect.

12.11 (*a*) In Example 12.7, does the force of static friction exerted on the rear wheel cause a torque on the wheel? If so, in which direction? If not, why not? (*b*) In what direction is the sum of the torques on the rear wheel? What does this tell you about the relative magnitudes of the individual torques on the wheel?

12.7 Torque and energy

Torques cause objects to accelerate rotationally and thus cause a change in their rotational kinetic energy. To calculate this energy change, consider the object shown in **Figure 12.41**. A force $\vec{F}$ is exerted at point P on the object, a distance r from the axis of rotation. Let this force be such that the magnitude of its component perpendicular to r remains constant as the object rotates. This component thus causes a constant torque $\tau_\vartheta = +rF_\perp$ about the axis. Equation 12.10 relates the sum of the torques to the rotational acceleration. By applying the chain rule (Appendix B), we can write this equation in the form

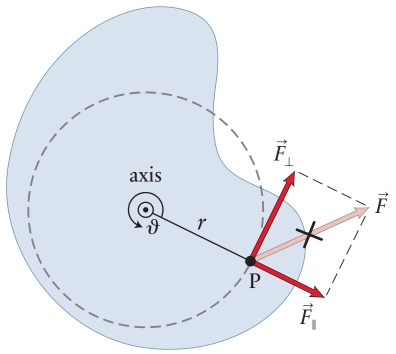


Figure 12.41 A rigid object subject to a constant torque caused by a force $\vec{F}$ exerted on it undergoes a rotational displacement $\Delta\vartheta$.

$$\Sigma\tau_{\text{ext}\,\vartheta} = I\alpha_\vartheta = I\frac{d\omega_\vartheta}{dt} = I\frac{d\omega_\vartheta}{d\vartheta}\frac{d\vartheta}{dt} = I\frac{d\omega_\vartheta}{d\vartheta}\omega_\vartheta, \qquad (12.27)$$

so

$$\left(\Sigma\tau_{\text{ext}\,\vartheta}\right)d\vartheta = I\omega_\vartheta\, d\omega_\vartheta. \qquad (12.28)$$

Integrating the left side of this equation yields

$$\int_{\vartheta_\text{i}}^{\vartheta_\text{f}}\left(\Sigma\tau_{\text{ext}\,\vartheta}\right)d\vartheta = \left(\Sigma\tau_{\text{ext}\,\vartheta}\right)\int_{\vartheta_\text{i}}^{\vartheta_\text{f}} d\vartheta = \left(\Sigma\tau_{\text{ext}\,\vartheta}\right)\Delta\vartheta. \qquad (12.29)$$

Integrating the right side after pulling the (constant) rotational inertia I out of the integral gives

$$\int_{\omega_{\vartheta,\text{i}}}^{\omega_{\vartheta,\text{f}}} I\omega_\vartheta\, d\omega_\vartheta = I\int_{\omega_{\vartheta,\text{i}}}^{\omega_{\vartheta,\text{f}}} \omega_\vartheta\, d\omega_\vartheta = I\left[\tfrac{1}{2}\omega_\vartheta^2\right]_{\omega_{\vartheta,\text{i}}}^{\omega_{\vartheta,\text{f}}}$$

$$= \tfrac{1}{2}I\omega_{\vartheta,\text{f}}^2 - \tfrac{1}{2}I\omega_{\vartheta,\text{i}}^2 = K_{\text{rot,f}} - K_{\text{rot,i}}. \qquad (12.30)$$

Equations 12.28–12.30 thus show that the change in the rotational kinetic energy, $\Delta K_\text{rot} \equiv K_{\text{rot,f}} - K_{\text{rot,i}}$, is

$$\Delta K_\text{rot} = \left(\Sigma\tau_{\text{ext}\,\vartheta}\right)\Delta\vartheta \quad \text{(constant torques, rigid object).} \qquad (12.31)$$

This equation states that the change in an object's rotational kinetic energy is equal to the product of the sum of the torques on the object and the object's rotational displacement. Note the analogy to Eq. 9.14, which relates the change in an object's center-of-mass kinetic energy to the product of the vector sum of the forces exerted on it and the object's displacement:

$$\Delta K_\text{cm} = \left(\Sigma F_\text{ext}\right)\Delta x_\text{cm} \quad \text{(constant forces, one dimension).} \qquad (12.32)$$

The kinetic energy of an object or system that is in both translational and rotational motion is equal to the sum of its center-of-mass and rotational kinetic energies:

$$K = K_\text{cm} + K_\text{rot} = \tfrac{1}{2}mv_\text{cm}^2 + \tfrac{1}{2}I\omega^2, \qquad (12.33)$$

and the change in this kinetic energy is given by the sum of the changes in Eqs. 12.31 and 12.32:

$$\Delta K = \Delta K_\text{cm} + \Delta K_\text{rot}. \qquad (12.34)$$

EXAMPLE 12.8 Rolling down a ramp

A solid cylindrical object of inertia m, rotational inertia I, and radius R rolls down a ramp that makes an angle θ with the horizontal. By how much does the cylinder's energy increase if it is released from rest and its center of mass drops a vertical distance h?

❶ **GETTING STARTED** I am given information about an object in the shape of a solid cylinder—inertia, rotational inertia, radius, and initial speed—and my task is to find out how much the object's energy has increased once it has rolled down a ramp such that its center of mass has traveled a vertical distance h. The object accelerates down the incline under the influence of the force of gravity. I therefore begin by making a sketch of the situation and drawing both free-body and extended free-body diagrams (**Figure 12.42**). The object is subject to a gravitational force exerted by Earth and a contact force exerted by the ramp. If I choose my axes as shown in my sketch, the contact force exerted by the ramp has a normal component $\vec{F}^{n}_{\text{ro}}$ in the y direction and a tangential component $\vec{F}^{s}_{\text{ro}}$ in the negative x direction due to static friction.

❷ **DEVISE PLAN** As the object rolls, both its translational and rotational kinetic energies increase. Because the shape of the object does not change and because static friction is nondissipative, the object's internal energy does not change. I can use Eq. 12.32 for the change in translational kinetic energy and Eq. 12.31 for the change in rotational kinetic energy. To express the two factors on the right in Eq. 12.32 in terms of my given variables, I use the geometry of the situation to express both factors in terms of $\sin\theta$.

❸ **EXECUTE PLAN** The change in translational kinetic energy is given by Eq. 12.32, $\Delta K_{\text{cm}} = \left(\sum F_{\text{ext}\,x}\right)\Delta x_{\text{cm}}$, and the vector sum of the forces exerted on the object in the x direction is

$$\sum F_{\text{ext}\,x} = mg\sin\theta - F^{s}_{\text{ro}}$$

or, using Eq. 12.26,

$$\sum F_{\text{ext}\,x} = +mg\sin\theta\left(1 - \frac{1}{1 + c^{-1}}\right),$$

where c is the object's shape factor. Because the displacement of the object's center of mass along the plane is $\Delta x_{\text{cm}} = h/\sin\theta$, the change in its translational kinetic energy is

$$\Delta K_{\text{cm}} = \left(\sum F_{\text{ext}\,x}\right)\Delta x_{\text{cm}} = mgh\left(1 - \frac{1}{1 + c^{-1}}\right).$$

Next, I use Eq. 12.31 to calculate the change in the object's rotational kinetic energy. From my extended free-body diagram I see that only the force of static friction causes a (positive) torque, so $\sum \tau_{\text{ext}\,\vartheta} = +F^{s}_{\text{ro}}R$. I can find the object's rotational displacement $\Delta\vartheta$ from Eq. 12.18:

$$\Delta\vartheta = \frac{\Delta x_{\text{cm}}}{R} = \left(\frac{h}{\sin\theta}\right)\left(\frac{1}{R}\right) = +\frac{h}{R\sin\theta},$$

so, from Eq. 12.31,

$$\Delta K_{\text{rot}} = (F^{s}_{\text{ro}}R)\left(\frac{h}{R\sin\theta}\right) = \left(\frac{mg\sin\theta}{1 + c^{-1}}\right)\left(\frac{h}{\sin\theta}\right) = \frac{mgh}{1 + c^{-1}},$$

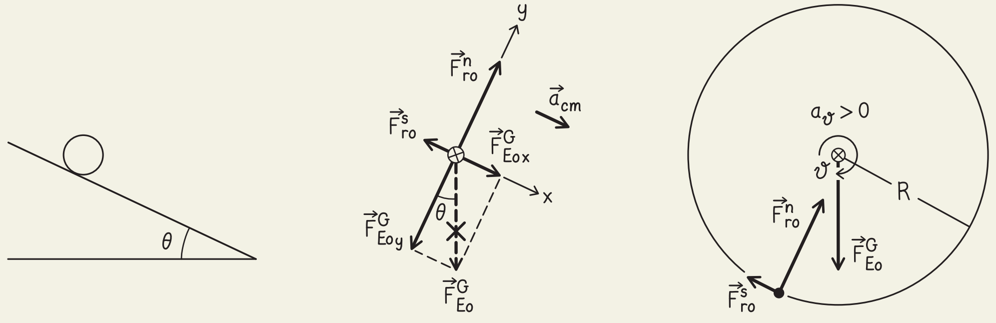
where I have again used Eq. 12.26 to substitute for F^{s}_{ro}. Adding the two changes in kinetic energy, I obtain

$$\Delta E = \Delta K_{\text{cm}} + \Delta K_{\text{rot}}$$

$$= mgh\left(1 - \frac{1}{1 + c^{-1}}\right) + mgh\left(\frac{1}{1 + c^{-1}}\right) = mgh. ✔$$

❹ **EVALUATE RESULT** My result indicates that the object's energy changes by the same amount it would change if it were simply in free fall! In other words, the only work done on the object is the work done by the gravitational force: $\vec{F}^{G}_{\text{Eo}} \cdot \Delta\vec{r} = mgh$ (see Section 10.9). This implies that the work done by all other forces on the object is zero. The normal force does no work on the object because it is perpendicular to the displacement of the object, but what is the work done by the force of static friction on the object? The object's displacement, $h/\sin\theta$, lies along the line of action of the force of static friction, and so it is tempting to write $-F^{s}_{\text{ro}}(h/\sin\theta)$ for the work done by the force of static friction on the object. However, the point of application for $\vec{F}^{s}_{\text{ro}}$ has zero velocity. At each instant, a different point on the object's surface touches the ramp, but the instantaneous velocity of that point is zero. The force displacement for $\vec{F}^{s}_{\text{ro}}$ is thus zero, so the work done by this force on the object is zero as well.

Figure 12.42



Example 12.8 shows that *the force of static friction does no work on a rolling object*! The only thing it does is take away some of the translational kinetic energy and generate an equal amount of rotational kinetic energy. It takes away translational kinetic energy by reducing the magnitude of the vector sum of the forces and converts it to rotational kinetic energy by causing a torque.

> **12.12** In Figure 12.41, does the force exerted on the rigid object do work on the object?

12.8 The vector product

To complete our treatment of rotation, we must examine the vectorial nature of the rotational components. As we saw in the first part of this chapter, an object's rotational velocity can be described by a vector $\vec{\omega}$. Angular momentum, which is the product of a scalar and $\vec{\omega}$, $\vec{L} = I\vec{\omega}$, is therefore also a vector and points in the same direction as $\vec{\omega}$. Because the rotational acceleration $\vec{\alpha}$ is the time derivative of $\vec{\omega}$, it must be a vector, too. Torque, which is the product of a scalar and $\vec{\alpha}$, $\vec{\tau} = I\vec{\alpha}$, therefore points in the same direction as $\vec{\alpha}$.

Consider the object in **Figure 12.43**. A force $\vec{F}$ exerted at point P increases the magnitude of the object's rotational velocity about the rotation axis at O. As the magnitude of the rotational velocity increases, the length of the vector representing $\vec{\omega}$ increases. Because the rotation takes place in a plane, the direction of $\vec{\omega}$ remains along the axis of rotation. The direction of $\vec{\alpha}$ is related to the direction of the vector representing the change in rotational velocity $\Delta\vec{\omega}$. Because the direction of $\vec{\omega}$ remains along the axis of rotation, $\Delta\vec{\omega}$, $\vec{\alpha}$, and $\vec{\tau}$ also lie along the rotation axis.

The magnitude of the torque is $rF\sin\theta$, where θ is the angle between $\vec{r}$ and $\vec{F}$. In Section 10.9, I showed that work on a system can be written as a scalar product of two vectors. To account for the vectorial nature of torque, we must introduce a new kind of product of two vectors: one that generates a new vector. Specifically, this product of the vectors $\vec{r}$ and $\vec{F}$ must generate a vector whose magnitude is $rF\sin\theta$ and whose direction is as indicated in **Figure 12.44**. Such a product is called the **vector product.** The vector product of two vectors $\vec{A}$ and $\vec{B}$, written $\vec{A}\times\vec{B}$, is a vector whose magnitude is

$$|\vec{A}\times\vec{B}| \equiv AB\sin\theta, \qquad (12.35)$$

where θ is the angle between $\vec{A}$ and $\vec{B}$ when they are placed tail to tail (and $\theta \le 180°$). The direction of the vector $\vec{A}\times\vec{B}$ is determined as illustrated in Figure 12.44*a*: Place vectors $\vec{A}$ and $\vec{B}$ with their tails together and apply the right-hand rule to the direction of rotation obtained by rotating $\vec{A}$ into $\vec{B}$

Figure 12.43 For a rotation about a fixed axis with increasing rotational speed, the rotational velocity, rotational acceleration, and torque vectors all point along the rotation axis.

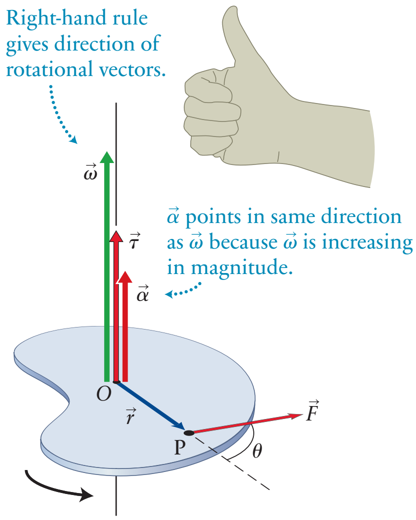


Figure 12.44 The vector product of two vectors.