


GLOBAL EDITION

# Digital Image Processing

FOURTH EDITION

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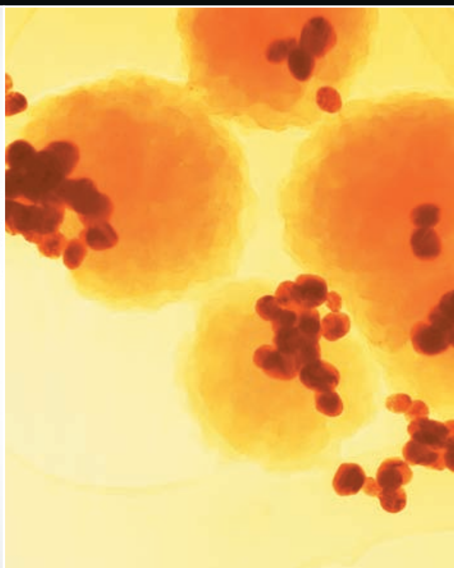

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(Brigham [1988]). We know from the previous section that 2-D DFTs can be implemented by successive passes of the 1-D transform, so we need to focus only on the FFT of one variable.

In derivations of the FFT, it is customary to express Eq. (4-44) in the form

$$F(u) = \sum_{x=0}^{M-1} f(x) W_M^{ux} \tag{4-159}$$

for $u = 0, 1, 2, \ldots, M-1$, where

$$W_M = e^{-j2\pi/M} \tag{4-160}$$

and $M$ is assumed to be of the form

$$M = 2^p \tag{4-161}$$

where $p$ is a positive integer. Then it follows that $M$ can be expressed as

$$M = 2K \tag{4-162}$$

with $K$ being a positive integer also. Substituting Eq. (4-162) into Eq. (4-159) yields

$$\begin{aligned} F(u) &= \sum_{x=0}^{2K-1} f(x) W_{2K}^{ux} \\ &= \sum_{x=0}^{K-1} f(2x) W_{2K}^{u(2x)} + \sum_{x=0}^{K-1} f(2x+1) W_{2K}^{u(2x+1)} \end{aligned} \tag{4-163}$$

However, it can be shown using Eq. (4-160) that $W_{2K}^{2ux} = W_K^{ux}$, so Eq. (4-163) can be written as

$$F(u) = \sum_{x=0}^{K-1} f(2x) W_K^{ux} + \sum_{x=0}^{K-1} f(2x+1) W_K^{ux} W_{2K}^{u} \tag{4-164}$$

Defining

$$F_{\text{even}}(u) = \sum_{x=0}^{K-1} f(2x) W_K^{ux} \tag{4-165}$$

for $u = 0, 1, 2, \ldots, K-1$, and

$$F_{\text{odd}}(u) = \sum_{x=0}^{K-1} f(2x+1) W_K^{ux} \tag{4-166}$$

for $u = 0, 1, 2, \ldots, K-1$, reduces Eq. (4-164) to

$$F(u) = F_{\text{even}}(u) + F_{\text{odd}}(u) W_{2K}^{u} \tag{4-167}$$

Also, because $W_M^{u+K} = W_K^u$ and $W_{2K}^{u+K} = -W_{2K}^u$, it follows that

$$F(u+K) = F_{\text{even}}(u) - F_{\text{odd}}(u)W_{2K}^u \tag{4-168}$$

Analysis of Eqs. (4-165) through (4-168) reveals some important (and surprising) properties of these expressions. An $M$-point DFT can be computed by dividing the original expression into two parts, as indicated in Eqs. (4-167) and (4-168). Computing the first half of $F(u)$ requires evaluation of the two $(M/2)$-point transforms given in Eqs. (4-165) and (4-166). The resulting values of $F_{\text{even}}(u)$ and $F_{\text{odd}}(u)$ are then substituted into Eq. (4-167) to obtain $F(u)$ for $u = 0, 1, 2, \ldots, (M/2 - 1)$. The other half then follows directly from Eq. (4-168) *without* additional transform evaluations.

It is of interest to examine the computational implications of the preceding procedure. Let $\mathfrak{m}(p)$ and $\mathfrak{a}(p)$ represent the number of complex multiplications and additions, respectively, required to implement the method. As before, the number of samples is $2^p$, where $p$ is a positive integer. Suppose first that $p = 1$ so that the number of samples is two. A two-point transform requires the evaluation of $F(0)$; then $F(1)$ follows from Eq. (4-168). To obtain $F(0)$ requires computing $F_{\text{even}}(0)$ and $F_{\text{odd}}(0)$. In this case $K = 1$ and Eqs. (4-165) and (4-166) are one-point transforms. However, because the DFT of a single sample point is the sample itself, no multiplications or additions are required to obtain $F_{\text{even}}(0)$ and $F_{\text{odd}}(0)$. One multiplication of $F_{\text{odd}}(0)$ by $W_2^0$ and one addition yields $F(0)$ from Eq. (4-167). Then $F(1)$ follows from Eq. (4-168) with one more addition (subtraction is considered to be the same as addition). Because $F_{\text{odd}}(0)W_2^0$ has been computed already, the total number of operations required for a two-point transform consists of $\mathfrak{m}(1) = 1$ multiplication and $\mathfrak{a}(1) = 2$ additions.

The next allowed value for $p$ is 2. According to the preceding development, a four-point transform can be divided into two parts. The first half of $F(u)$ requires evaluation of two, two-point transforms, as given in Eqs. (4-165) and (4-166) for $K = 2$. A two-point transform requires $\mathfrak{m}(1)$ multiplications and $\mathfrak{a}(1)$ additions. Therefore, evaluation of these two equations requires a total of $2\mathfrak{m}(1)$ multiplications and $2\mathfrak{a}(1)$ additions. Two further multiplications and additions are necessary to obtain $F(0)$ and $F(1)$ from Eq. (4-167). Because $F_{\text{odd}}(u)W_{2K}^u$ has been computed already for $u = \{0,1\}$, two more additions give $F(2)$ and $F(3)$. The total is then $\mathfrak{m}(2) = 2\mathfrak{m}(1) + 2$ and $\mathfrak{a}(2) = 2\mathfrak{a}(1) + 4$.

When $p$ is equal to 3, two four-point transforms are needed to evaluate $F_{\text{even}}(u)$ and $F_{\text{odd}}(u)$. They require $2\mathfrak{m}(2)$ multiplications and $2\mathfrak{a}(2)$ additions. Four more multiplications and eight more additions yield the complete transform. The total then is then $\mathfrak{m}(3) = 2\mathfrak{m}(2) + 4$ multiplication and $\mathfrak{a}(3) = 2\mathfrak{a}(2) + 8$ additions.

Continuing this argument for any positive integer $p$ leads to recursive expressions for the number of multiplications and additions required to implement the FFT:

$$\mathfrak{m}(p) = 2\mathfrak{m}(p-1) + 2^{p-1} \qquad p \geq 1 \tag{4-169}$$

and

$$\mathfrak{a}(p) = 2\mathfrak{a}(p-1) + 2^p \qquad p \geq 1 \tag{4-170}$$

where $\mathfrak{m}(0) = 0$ and $\mathfrak{a}(0) = 0$ because the transform of a single point does not require any multiplication or additions.

The method just developed is called the *successive doubling FFT algorithm* because it is based on computing a two-point transform from two one-point transforms, a four-point transform from two two-point transforms, and so on, for any $M$ equal to an integer power of 2. It is left as an exercise (see Problem 4.63) to show that

$$\mathfrak{m}(p) = \frac{1}{2} M \log_2 M \tag{4-171}$$

and

$$\mathfrak{a}(n) = M \log_2 M \tag{4-172}$$

where $M = 2^p$.

The computational advantage of the FFT over a direct implementation of the 1-D DFT is defined as

$$\begin{aligned} C(M) &= \frac{M^2}{M \log_2 M} \\ &= \frac{M}{\log_2 M} \end{aligned} \tag{4-173}$$

where $M^2$ is the number of operations required for a "brute force" implementation of the 1-D DFT. Because it is assumed that $M = 2^p$, we can write Eq. (4-173) in terms of $p$:

$$C(p) = \frac{2^p}{p} \tag{4-174}$$

A plot of this function (Fig. 4.67) shows that the computational advantage increases rapidly as a function of $p$. For example, when $p = 15$ (32,768 points), the FFT has nearly a 2,200 to 1 advantage over a brute-force implementation of the DFT. Thus, we would expect that the FFT can be computed nearly 2,200 times faster than the DFT on the same machine. As you learned in Section 4.1, the FFT also offers significant computational advantages over spatial filtering, with the cross-over between the two approaches being for relatively small kernels.

There are many excellent sources that cover details of the FFT so we will not dwell on this topic further (see, for example, Brigham [1988]). Most comprehensive signal and image processing software packages contain generalized implementations of the FFT that do not require the number of points to be an integer power

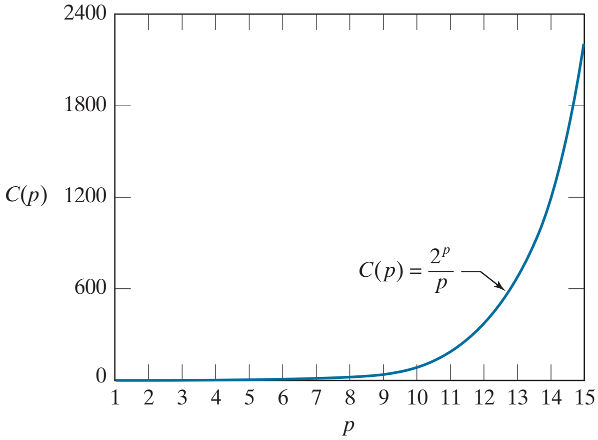



**FIGURE 4.67** Computational advantage of the FFT over a direct implementation of the 1-D DFT. The number of samples is $M = 2^p$. The computational advantage increases rapidly as a function of $p$.

of 2 (at the expense of slightly less efficient computation). Free FFT programs also are readily available, principally over the internet.

## Summary, References, and Further Reading

The material in this chapter is a progression from sampling to the Fourier transform, and then to filtering in the frequency domain. Some of the concepts, such as the sampling theorem, make very little sense if not explained in the context of the frequency domain. The same is true of effects such as aliasing. Thus, the material developed in the preceding sections is a solid foundation for understanding the fundamentals of 2-D digital signal processing. We took special care to develop the material starting with basic principles, so that any reader with a modest mathematical background would be in a position not only to absorb the material, but also to apply it.

For complementary reading on the 1-D and 2-D continuous Fourier transforms, see the books by Bracewell [1995, 2003]. These two books, together with Castleman [1996], Petrou and Petrou [2010], Brigham [1988], and Smith [2003], provide additional background for the material in Sections 4.2 through 4.6. Sampling phenomena such as aliasing and moiré patterns are topics amply illustrated in books on computer graphics, as exemplified by Hughes and Andries [2013]. For additional general background on the material in Sections 4.7 through 4.11 see Hall [1979], Jain [1989], Castleman [1996], and Pratt [2014]. For details on the software aspects of many of the examples in this chapter, see Gonzalez, Woods, and Eddins [2009].

## Problems

*Solutions to the problems marked with an asterisk (*) are in the DIP4E Student Support Package (consult the book website: www.ImageProcessingPlace.com)*

**4.1** Answer the following:

**(a)*** Give an equation similar to Eq. (4-10), but for an impulse located at $t = t_0$.

**(b)** Repeat for Eq. (4-15).

**(c)*** Is it correct to say that $\delta(t - a) = \delta(a - t)$ in general? Explain.

**4.2** Repeat Example 4.1, but using the function $f(t) = A$ for $0 \le t < T$ and $f(t) = 0$ for all other values of $t$. Explain the reason for any differences between your results and the results in the example.

**4.3** What is the convolution of two, 1-D impulses:

**(a)*** $\delta(t)$ and $\delta(t - t_0)$?

**(b)** $\delta(t - t_0)$ and $\delta(t + t_0)$?

**4.4*** Use the sifting property of the impulse to show that convolving a 1-D continuous function, $f(t)$, with an impulse located at $t_0$ shifts the function so that its origin is moved to the location of the impulse (if the impulse is at the origin, the function is not shifted).

**4.5*** With reference to Fig. 4.9, give a graphical illustration of an aliased pair of functions that are not periodic.

**4.6** With reference to Fig. 4.11:

**(a)*** Redraw the figure, showing what the dots would look like for a sampling rate that exceeds the Nyquist rate slightly.

**(b)** What is the *approximate* sampling rate represented by the large dots in Fig. 4.11?

**(c)** *Approximately*, what would be the lowest sampling rate that you would use so that (1) the Nyquist rate is satisfied, and (2) the samples look like a sine wave?

**4.7** A function, $f(t)$, is formed by the sum of three functions, $f_1(t) = A\sin(\pi t)$, $f_2(t) = B\sin(4\pi t)$, and $f_3(t) = C\cos(8\pi t)$.

**(a)** Assuming that the functions extend to infinity in both directions, what is the highest frequency of $f(t)$? (*Hint:* Start by finding the period of the sum of the three functions.)

**(b)*** What is the Nyquist rate corresponding to your result in (a)? (Give a numerical answer.)

**(c)** At what rate would you sample $f(t)$ so that perfect recovery of the function from its samples is possible?

**4.8*** Show that $\Im\{e^{j2\pi t_0 t}\} = \delta(\mu - t_0)$, where $t_0$ is a constant. (*Hint:* Study Example 4.2.)

**4.9** Show that the following expressions are true. (*Hint:* Make use of the solution to Problem 4.8):

**(a)*** $\Im\{\cos(2\pi\mu_0 t)\} = \frac{1}{2}[\delta(\mu - \mu_0) + \delta(\mu + \mu_0)]$

**(b)** $\Im\{\sin(2\pi\mu_0 t)\} = \frac{1}{2j}[\delta(\mu - \mu_0) - \delta(\mu + \mu_0)]$

**4.10** Consider the function $f(t) = \sin(2\pi n t)$, where $n$ is an integer. Its Fourier transform, $F(\mu)$, is purely imaginary (see Problem 4.9). Because the transform, $\tilde{F}(\mu)$, of sampled data consists of periodic copies of $F(\mu)$, it follows that $\tilde{F}(\mu)$ will also be purely imaginary. Draw a diagram similar to Fig. 4.6, and answer the following questions based on your diagram (assume that sampling starts at $t = 0$).

**(a)*** What is the period of $f(t)$?

**(b)*** What is the frequency of $f(t)$?

**(c)*** What would the sampled function and its Fourier transform look like in general if $f(t)$ is sampled at a rate higher than the Nyquist rate?

**(d)** What would the sampled function look like in general if $f(t)$ is sampled at a rate lower than the Nyquist rate?

**(e)** What would the sampled function look like if $f(t)$ is sampled at the Nyquist rate, with samples taken at $t = 0, \pm\Delta T, \pm 2\Delta T, \ldots$ ?

**4.11*** Prove the validity of the convolution theorem of one continuous variable, as given in Eqs. (4-25) and (4-26).

**4.12** We explained in the paragraph after Eq. (4-36) that arbitrarily limiting the duration of a band-limited function by multiplying it by a box function would cause the function to cease being band limited. Show graphically why this is so by limiting the duration of the function $f(t) = \cos(2\pi\mu_0 t)$ [the Fourier transform of this function is given in Problem 4.9(a)]. (*Hint:* The transform of a box function is given in Example 4.1. Use that result in your solution, and also the fact that convolution of a function with an impulse shifts the function to the location of the impulse, in the sense discussed in the solution of Problem 4.4.)

**4.13*** Complete the steps that led from Eq. (4-37) to Eq. (4-38).

**4.14** Show that $\tilde{F}(\mu)$ in Eq. (4-40) is infinitely periodic in both directions, with period $1/\Delta T$.

**4.15** Do the following:

**(a)** Show that Eqs. (4-42) and (4-43) are a Fourier transform pair: $f_n \Leftrightarrow F_m$.

**(b)*** Show that Eqs. (4-44) and (4-45) also are a Fourier transform pair: $f(x) \Leftrightarrow F(u)$.

You will need the following orthogonality property in both parts of this problem:

$$\sum_{x=0}^{M-1} e^{j2\pi rx/M} e^{-j2\pi ux/M} = \begin{cases} M & \text{if } r = u \\ 0 & \text{otherwise} \end{cases}$$

**4.16** Show that both $F(u)$ and $f(x)$ in Eqs. (4-44) and (4-45) are infinitely periodic with period $M$; that is, $F(u) = F(u + kM)$ and $f(x) = f(x + M)$, where $k$ is an integer. [See Eqs. (4-46) and (4-47).]

**4.17** Demonstrate the validity of the translation (shift) properties of the following 1-D, discrete Fourier transform pairs. (*Hint:* It is easier in part (b) to work with the IDFT.)

**(a)*** $f(x)e^{j2\pi u_0 x/M} \Leftrightarrow F(u - u_0)$

**(b)** $f(x - x_0) \Leftrightarrow F(u)e^{-j2\pi u x_0/M}$

**4.18** Show that the 1-D convolution theorem given in Eqs. (4-25) and (4-26) also holds for discrete variables, but with the right side of Eq. (4-26) multiplied by $1/M$. That is, show that

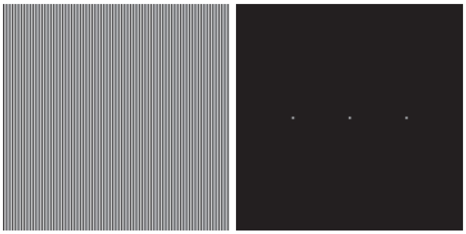
**(a)*** $(f \star h)(x) \Leftrightarrow (F \bullet H)(u)$, and

**(b)** $(f \bullet h)(x) \Leftrightarrow \dfrac{1}{M}(F \star H)(u)$

**4.19*** Extend the expression for 1-D convolution [see Eq. (4-24)] to two continuous variables. Use $t$ and $z$ for the variables on the left side of the expression and $\alpha$ and $\beta$ for the variables in the 2-D integral.

**4.20** Use the sifting property of the 2-D impulse to show that convolution of a 2-D continuous function, $f(t,z)$, with an impulse shifts the function so that its origin is located at the location of the impulse. (If the impulse is at the origin, the function is copied exactly as it was.) (*Hint:* Study the solution to Problem 4.4).

**4.21** The image on the left in the figure below consists of alternating stripes of black/white, each stripe



being two pixels wide. The image on the right is the Fourier spectrum of the image on the left, showing the dc term and the frequency terms corresponding to the stripes. (Remember, the spectrum is symmetric so all components, other than the dc term, appear in two symmetric locations.)

**(a)*** Suppose that the stripes of an image of the same size are four pixels wide. Sketch what the spectrum of the image would look like, including only the dc term and the two highest-value frequency terms, which correspond to the two spikes in the spectrum above.

**(b)** Why are the components of the spectrum limited to the horizontal axis?

**(c)** What would the spectrum look like for an image of the same size but having stripes that are one pixel wide? Explain the reason for your answer.

**(d)** Are the dc terms in (a) and (c) the same, or are they different? Explain.

**4.22** A high-technology company specializes in developing imaging systems for digitizing images of commercial cloth. The company has a new order for 1,000 systems for digitizing cloth consisting of repeating black and white vertical stripes, each of width 2 cm. Optical and mechanical engineers have already designed the front-end optics and mechanical positioning mechanisms so that you are guaranteed that every image your system digitizes starts with a complete black vertical stripe and ends with a complete white stripe. Every image acquired will contain exactly 250 vertical stripes. Noise and optical distortions are negligible. Having learned of your success in taking an image processing course, the company employs you to specify the resolution of the imaging chip to be used in the new system. The optics can be adjusted to project the field of view accurately onto the area defined by the size of the chip you specify. Your design will be implemented in hundreds of locations, so cost is an important consideration. What resolution chip (in terms of number of imaging elements per horizontal line) would you specify to avoid aliasing?

**4.23*** We know from the discussion in Section 4.5 that zooming or shrinking a digital image generally causes aliasing. Give an example of an image that would be free of aliasing if it were zoomed by pixel replication.

**4.24** With reference to the discussion on linearity in Section 2.6, demonstrate that

**(a)*** The 2-D continuous Fourier transform is a linear operator.

**(b)** The 2-D DFT is a linear operator also.