GLOBAL EDITION

Fundamentals of Applied Electromagnetics

EIGHTH EDITION

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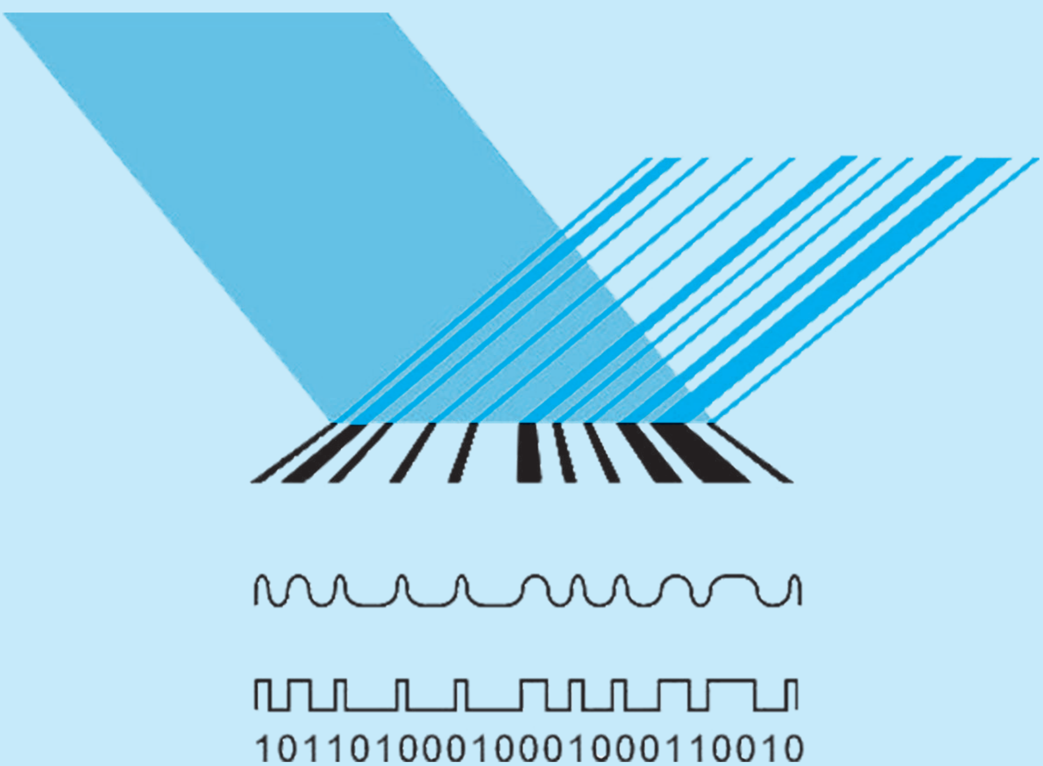

FUNDAMENTALS OF APPLIED ELECTROMAGNETICS

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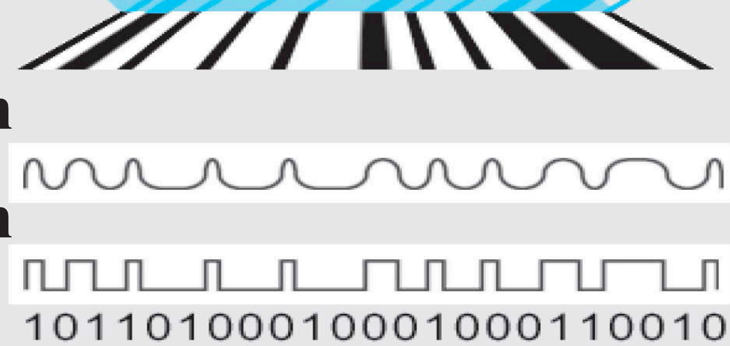

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directions of the three mutually perpendicular unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, which are also called ***base vectors***. The vector **A** in **Fig. 3-2(b)** may be decomposed as

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.3}$$

where A_x, A_y, and A_z are **A**'s scalar components along the x-, y-, and z axes, respectively. The component A_z is equal to the perpendicular projection of **A** onto the z axis, and similar definitions apply to A_x and A_y. Application of the Pythagorean theorem—first to the right triangle in the x–y plane to express the hypotenuse A_r in terms of A_x and A_y and then again to the vertical right triangle with sides A_r and A_z and hypotenuse A—yields the following expression for the magnitude of **A**:

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}. \tag{3.4}$$

Since A is a nonnegative scalar, only the positive root applies. From Eq. (3.2), the unit vector $\hat{\mathbf{a}}$ is

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}. \tag{3.5}$$

Occasionally, we use the shorthand notation $\mathbf{A} = (A_x, A_y, A_z)$ to denote a vector with components A_x, A_y, and A_z in a Cartesian coordinate system.

3-1.1 Equality of Two Vectors

Two vectors **A** and **B** are equal if they have equal magnitudes and identical unit vectors. Thus, if

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.6a}$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \tag{3.6b}$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

▶ Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another. ◀

3-1.2 Vector Addition and Subtraction

The sum of two vectors **A** and **B** is a vector

$$\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z,$$

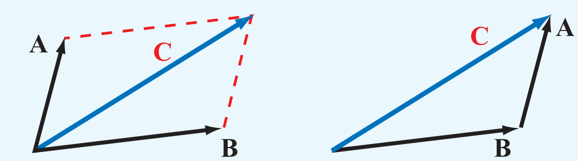

Figure 3-3 Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

given by

$$\begin{aligned}\mathbf{C} = \mathbf{A} + \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) + (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z) \\ &= \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y + \hat{\mathbf{z}}C_z,\end{aligned} \tag{3.7}$$

with $C_x = A_x + B_x$, etc.

▶ Vector addition is commutative:

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{3.8}$$

Graphically, vector addition can be accomplished by either the parallelogram or the head-to-tail rule (**Fig. 3-3**). Vector **C** is the diagonal of the parallelogram with sides **A** and **B**. With the head-to-tail rule, we may either add **A** to **B** or **B** to **A**. When **A** is added to **B**, it is repositioned so that its tail starts at the tip of **B** while keeping its length and direction unchanged. The sum vector **C** starts at the tail of **B** and ends at the tip of **A**.

Subtraction of vector **B** from vector **A** is equivalent to the addition of **A** to negative **B**. Thus,

$$\begin{aligned}\mathbf{D} = \mathbf{A} - \mathbf{B} &= \mathbf{A} + (-\mathbf{B}) \\ &= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z).\end{aligned} \tag{3.9}$$

Graphically, the same rules used for vector addition are also applicable to vector subtraction; the only difference is that the arrowhead of $(-\mathbf{B})$ is drawn on the opposite end of the line segment representing the vector **B** (i.e., the tail and head are interchanged).

3-1.3 Position and Distance Vectors

The ***position vector*** of a point P in space is the vector from the origin to P. Assuming points P_1 and P_2 are at (x_1, y_1, z_1) and (x_2, y_2, z_2) in **Fig. 3-4**, their position vectors are

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1, \tag{3.10a}$$

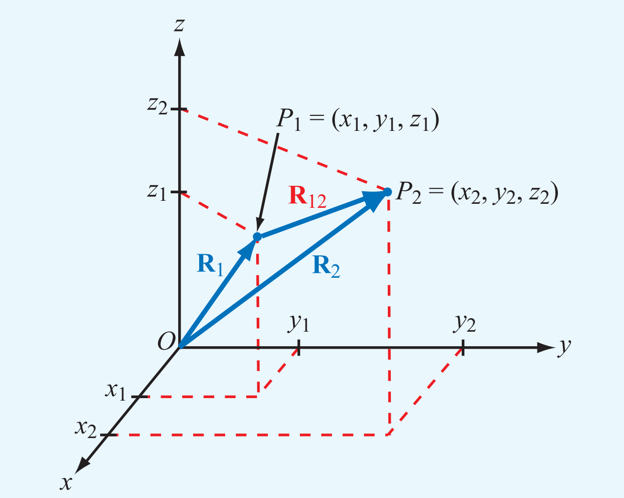

Figure 3-4 Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2, \tag{3.10b}$$

where point O is the origin.

The ***distance vector*** from P_1 to P_2 is defined as

$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1),\end{aligned} \tag{3.11}$$

and the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \tag{3.12}$$

Note that the first and second subscripts of $\mathbf{R}_{12}$ denote the locations of its tail and head, respectively (**Fig. 3-4**).

3-1.4 Vector Multiplication

There exist three types of products in vector calculus: the simple product, the scalar (or dot) product, and the vector (or cross) product.

Simple Product

The multiplication of a vector by a scalar is called a ***simple product***. The product of the vector $\mathbf{A} = \hat{\mathbf{a}}A$ by a scalar k results in a vector $\mathbf{B}$ with magnitude $B = kA$ and direction the same as $\mathbf{A}$. That is, $\hat{\mathbf{b}} = \hat{\mathbf{a}}$. In Cartesian coordinates,

$$\begin{aligned}\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}kA &= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z) \\ &= \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z.\end{aligned} \tag{3.13}$$

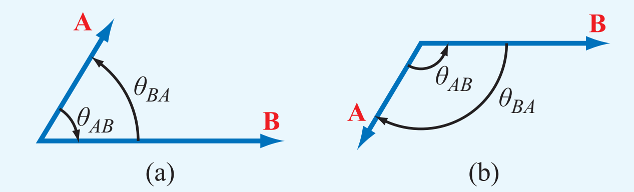

Figure 3-5 The angle θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$, measured from $\mathbf{A}$ to $\mathbf{B}$ between vector tails. The dot product is positive if $0 \leq \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \leq 180°$, as in (b).

Scalar or Dot Product

The ***scalar*** (or ***dot***) ***product*** of two co-anchored vectors $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} \cdot \mathbf{B}$ and pronounced "*A* dot *B*," is defined geometrically as the product of the magnitude of $\mathbf{A}$ and the scalar component of $\mathbf{B}$ along $\mathbf{A}$, or vice versa. Thus,

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}, \tag{3.14}$$

where θ_{AB} is the angle between $\mathbf{A}$ and $\mathbf{B}$ (**Fig. 3-5**) measured *from* the tail of $\mathbf{A}$ *to* the tail of $\mathbf{B}$. Angle θ_{AB} is assumed to be in the range $0 \leq \theta_{AB} \leq 180°$. The scalar product of $\mathbf{A}$ and $\mathbf{B}$ yields a scalar whose magnitude is less than or equal to the products of their magnitudes (equality holds when $\theta_{AB} = 0$) and whose sign is positive if $0 < \theta_{AB} < 90°$ and negative if $90° < \theta_{AB} < 180°$. When $\theta_{AB} = 90°$, $\mathbf{A}$ and $\mathbf{B}$ are orthogonal, and their dot product is zero. The quantity $A\cos\theta_{AB}$ is the scalar component of $\mathbf{A}$ along $\mathbf{B}$. Similarly, $B\cos\theta_{BA}$ is the scalar component of $\mathbf{B}$ along $\mathbf{A}$.

The dot product obeys both the commutative and distributive properties of multiplication:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}, \tag{3.15a}$$

(commutative property)

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}. \tag{3.15b}$$

(distributive property)

The commutative property follows from Eq. (3.14) and the fact that $\theta_{AB} = \theta_{BA}$. The distributive property expresses the fact that the scalar component of the sum of two vectors along a third one equals the sum of their respective scalar components.

The dot product of a vector with itself gives

$$\mathbf{A} \cdot \mathbf{A} = |\mathbf{A}|^2 = A^2, \tag{3.16}$$

which implies that

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}. \tag{3.17}$$

Also, θ_{AB} can be determined from

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\ \sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]. \tag{3.18}$$

Since the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are each orthogonal to the other two, it follows that

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \tag{3.19a}$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0. \tag{3.19b}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z). \tag{3.20}$$

Use of Eqs. (3.19a) and (3.19b) in Eq. (3.20) leads to

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z. \tag{3.21}$$

Vector or Cross Product

The ***vector*** (or ***cross***) ***product*** of two vectors **A** and **B**, denoted $\mathbf{A} \times \mathbf{B}$ and pronounced "*A* cross *B*," yields a vector defined as

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}, \tag{3.22}$$

where $\hat{\mathbf{n}}$ is a *unit vector normal to the plane containing* **A** *and* **B** (**Fig. 3-6(a)**). The magnitude of the cross product, $AB|\sin\theta_{AB}|$, equals the area of the parallelogram defined by the two vectors. The direction of $\hat{\mathbf{n}}$ is governed by the ***right-hand rule*** (**Fig. 3-6(b)**): $\hat{\mathbf{n}}$ points in the direction of the right thumb when the fingers rotate from **A** to **B** through the angle θ_{AB}. Note that, since $\hat{\mathbf{n}}$ is perpendicular to the plane containing **A** and **B**, $\mathbf{A} \times \mathbf{B}$ is perpendicular to both vectors **A** and **B**.

The cross product is anticommutative and distributive:

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \quad \textbf{(anticommutative)}. \tag{3.23a}$$

The anticommutative property follows from the application of the right-hand rule to determine $\hat{\mathbf{n}}$. The distributive property follows from the fact that the area of the parallelogram formed by **A** and $(\mathbf{B} + \mathbf{C})$ equals the sum of those formed by (**A** and **B**) and (**A** and **C**):

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}, \quad \textbf{(distributive)} \tag{3.23b}$$

The cross product of a vector with itself vanishes. That is,

$$\mathbf{A} \times \mathbf{A} = 0. \tag{3.24}$$

From the definition of the cross product given by Eq. (3.22), it is easy to verify that the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ of the Cartesian coordinate system obey the right-hand cyclic relations:

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \tag{3.25}$$

Note the cyclic order $(xyzxyz\ldots)$. Also,

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \tag{3.26}$$

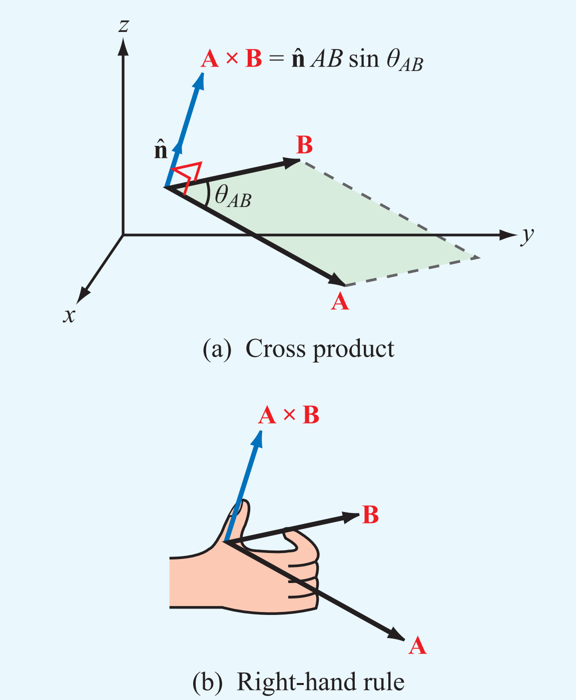

Figure 3-6 Cross product $\mathbf{A} \times \mathbf{B}$ points in the direction $\hat{\mathbf{n}}$, which is perpendicular to the plane containing **A** and **B** and defined by the right-hand rule.

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then use of Eqs. (3.25) and (3.26) leads to

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \times (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) \\ &\quad + \hat{\mathbf{z}}(A_xB_y - A_yB_x). \end{aligned} \tag{3.27}$$

The cyclical form of the result given by Eq. (3.27) allows us to express the cross product in the form of a determinant:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{3.28}$$

Example 3-1: Vectors and Angles

In Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** is directed from P_1 to point $P_2 = (1, -2, 2)$. Find:

(a) vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$,
(b) the angle between **A** and the y axis,
(c) vector **B**,
(d) the angle θ_{AB} between **A** and **B**, and
(e) perpendicular distance from the origin to vector **B**.

Solution: **(a)** Vector **A** is given by the position vector of $P_1 = (2, 3, 3)$ (**Fig. 3-7**). Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

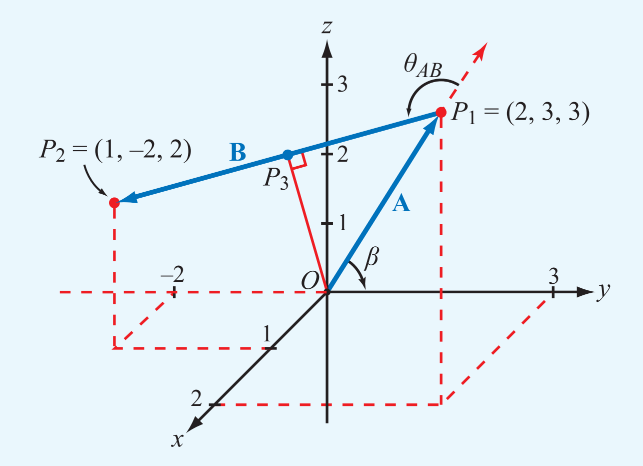

Figure 3-7 Geometry of Example 3-1.

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22},$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}.$$

(b) The angle β between **A** and the y axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ.$$

(c)

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\,\sqrt{27}}\right] = 145.1^\circ.$$

(e) The perpendicular distance between the origin and vector **B** is the distance $|\overrightarrow{OP_3}|$ shown in **Fig. 3-7**. From right triangle OP_1P_3,

$$\begin{aligned}|\overrightarrow{OP_3}| &= |\mathbf{A}| \sin(180^\circ - \theta_{AB}) \\ &= \sqrt{22}\sin(180^\circ - 145.1^\circ) = 2.68.\end{aligned}$$

Example 3-2: Cross Product

Given vectors $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3$ and $\mathbf{B} = \hat{\mathbf{y}}2 - \hat{\mathbf{z}}3$, compute (a) $\mathbf{A} \times \mathbf{B}$, (b) $\hat{\mathbf{y}} \times \mathbf{B}$, and (c) $(\hat{\mathbf{y}} \times \mathbf{B}) \cdot \mathbf{A}$.

Solution: **(a)** Application of Eq. (3.28) gives

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 2 & -1 & 3 \\ 0 & 2 & -3 \end{vmatrix} \\ &= \hat{\mathbf{x}}((-1)\times(-3) - 3\times 2) - \hat{\mathbf{y}}(2\times(-3) - 3\times 0) \\ &\quad + \hat{\mathbf{z}}(2\times 2 - (-1\times 0)) \\ &= -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}6 + \hat{\mathbf{z}}4.\end{aligned}$$

(b) $\hat{\mathbf{y}} \times \mathbf{B} = \hat{\mathbf{y}} \times (\hat{\mathbf{y}}2 - \hat{\mathbf{z}}3) = -\hat{\mathbf{x}}3.$

(c) $(\hat{\mathbf{y}} \times \mathbf{B}) \cdot \mathbf{A} = -\hat{\mathbf{x}}3 \cdot (\hat{\mathbf{x}}2 - \hat{\mathbf{y}} + \hat{\mathbf{z}}3) = -6.$

Exercise 3-1: Find the distance vector between $P_1 = (1, 2, 3)$ and $P_2 = (-1, -2, 3)$ in Cartesian coordinates.

Answer: $\overrightarrow{P_1P_2} = -\hat{\mathbf{x}}2 - \hat{\mathbf{y}}4$. (See EM.)

Exercise 3-2: Find the angle θ_{AB} between vectors **A** and **B** of Example 3-1 from the cross product between them.

Answer: $\theta_{AB} = 145.1°$. (See EM.)

Exercise 3-3: Find the angle between vector **B** of Example 3-1 and the z axis.

Answer: 101.1°. (See EM.)

Exercise 3-4: Vectors **A** and **B** lie in the y–z plane and both have the same magnitude of 2 (**Fig. E3.4**). Determine (a) $\mathbf{A} \cdot \mathbf{B}$ and (b) $\mathbf{A} \times \mathbf{B}$.

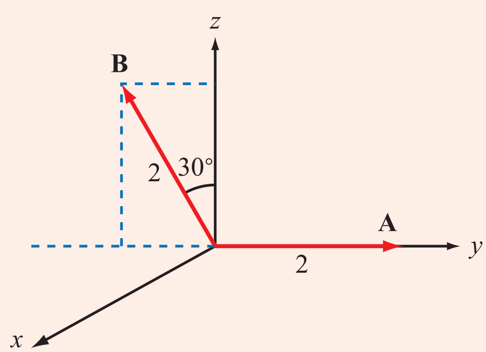

Figure E3.4

Answer: (a) $\mathbf{A} \cdot \mathbf{B} = -2$; (b) $\mathbf{A} \times \mathbf{B} = \hat{\mathbf{x}}3.46$. (See EM.)

Exercise 3-5: If $\mathbf{A} \cdot \mathbf{B} = \mathbf{A} \cdot \mathbf{C}$, does it follow that $\mathbf{B} = \mathbf{C}$?

Answer: No. (See EM.)

3-1.5 Scalar and Vector Triple Products

When three vectors are multiplied, not all combinations of dot and cross products are meaningful. For example, the product

$$\mathbf{A} \times (\mathbf{B} \cdot \mathbf{C})$$

does not make sense because $\mathbf{B} \cdot \mathbf{C}$ is a scalar, and the cross product of the vector **A** with a scalar is not defined under the rules of vector algebra. Other than the product of the form $\mathbf{A}(\mathbf{B} \cdot \mathbf{C})$, the only two meaningful products of three vectors are the scalar triple product and the vector triple product.

Scalar Triple Product

The dot product of a vector with the cross product of two other vectors is called a scalar triple product, so named because the result is a scalar. A scalar triple product obeys the cyclic order:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}). \quad (3.29)$$

The equalities hold as long as the cyclic order ($ABCABC\ldots$) is preserved. The scalar triple product of vectors $\mathbf{A} = (A_x, A_y, A_z)$, $\mathbf{B} = (B_x, B_y, B_z)$, and $\mathbf{C} = (C_x, C_y, C_z)$ can be expressed in the form of a 3×3 determinant:

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}. \quad (3.30)$$

The validity of Eqs. (3.29) and (3.30) can be verified by expanding **A**, **B**, and **C** in component form and carrying out the multiplications.

Vector Triple Product

The vector triple product involves the cross product of a vector with the cross product of two others, such as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}). \quad (3.31)$$

Since each cross product yields a vector, the result of a vector triple product is also a vector. The vector triple product does not obey the associative law. That is,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}, \quad (3.32)$$

which means that it is important to specify which cross multiplication is to be performed first. By expanding the vectors **A**, **B**, and **C** in component form, it can be shown that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B}), \quad (3.33)$$

which is known as the "bac-cab" rule.

Example 3-3: Vector Triple Product

Given $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}}2 + \hat{\mathbf{z}}3$, find $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ and compare it with $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

Solution:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

and

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}7 - \hat{\mathbf{z}}2.$$

A similar procedure gives $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}$. The fact that the results of two vector triple products are different demonstrates the inequality stated in Eq. (3.32).

Concept Question 3-1: When are two vectors equal and when are they identical?

Concept Question 3-2: When is the position vector of a point identical to the distance vector between two points?

Concept Question 3-3: If $\mathbf{A}\cdot\mathbf{B} = 0$, what is θ_{AB}?

Concept Question 3-4: If $\mathbf{A}\times\mathbf{B} = 0$, what is θ_{AB}?

Concept Question 3-5: Is $\mathbf{A}(\mathbf{B}\cdot\mathbf{C})$ a vector triple product?

Concept Question 3-6: If $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{C}$, does it follow that $\mathbf{B} = \mathbf{C}$?

3-2 Orthogonal Coordinate Systems

A three-dimensional coordinate system allows us to uniquely specify locations of points in space and the magnitudes and directions of vectors. Coordinate systems may be orthogonal or nonorthogonal.

► An ***orthogonal coordinate system*** is one in which coordinates are measured along locally mutually perpendicular axes. ◄

Nonorthogonal systems are very specialized and seldom used in solving practical problems. Many orthogonal coordinate systems have been devised, but the most commonly used are

- the Cartesian (also called rectangular),
- the cylindrical, and
- the spherical coordinate system.

Why do we need more than one coordinate system? Whereas a point in space has the same location and an object has the same shape regardless of which coordinate system is used to describe them, the solution of a practical problem can be greatly facilitated by the choice of a coordinate system that best fits the geometry under consideration. The following subsections examine the properties of each of the aforementioned orthogonal systems, and Section 3-3 describes how a point or vector may be transformed from one system to another.

3-2.1 Cartesian Coordinates

The Cartesian coordinate system was introduced in Section 3-1 to illustrate the laws of vector algebra. Instead of repeating these laws for the Cartesian system, we summarize them in Table 3-1. Differential calculus involves the use of differential lengths, areas, and volumes. In Cartesian coordinates, a ***differential length vector*** (Fig. 3-8) is expressed as

$$d\mathbf{l} = \hat{\mathbf{x}}\,dl_x + \hat{\mathbf{y}}\,dl_y + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz, \qquad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.

A ***differential area vector*** $d\mathbf{s}$ is a vector with magnitude ds equal to the product of two differential lengths (such as dl_y and dl_z) and direction specified by a unit vector along the third

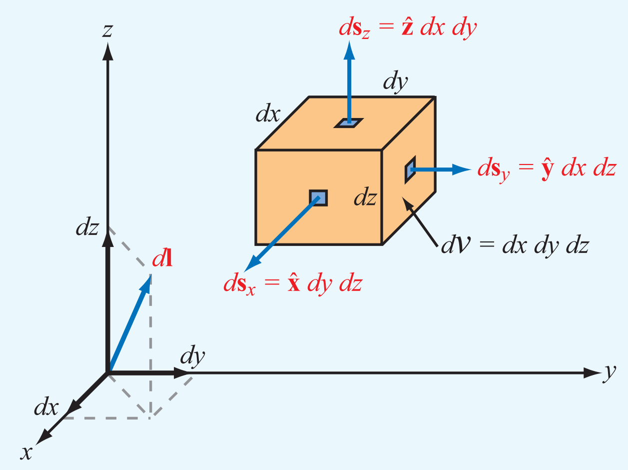

Figure 3-8 Differential length, area, and volume in Cartesian coordinates.