

CONTENTS
OF
THE ELEMENTS OF MATHEMATICS SERIES

I. THEORY OF SETS

1. Description of formal mathematics. 2. Theory of sets. 3. Ordered sets; cardinals; natural numbers. 4. Structures.

II. ALGEBRA

1. Algebraic structures. 2. Linear algebra. 3. Tensor algebras, exterior algebras, symmetric algebras. 4. Polynomials and rational fractions. 5. Fields. 6. Ordered groups and fields. 7. Modules over principal ideal rings. 8. Semi-simple modules and rings. 9. Sesquilinear and quadratic forms.

III. GENERAL TOPOLOGY

1. Topological structures. 2. Uniform structures. 3. Topological groups. 4. Real numbers. 5. One-parameter groups. 6. Real number spaces, affine and projective spaces. 7. The additive groups $\mathbf{R}^n$. 8. Complex numbers. 9. Use of real numbers in general topology. 10. Function spaces.

IV. FUNCTIONS OF A REAL VARIABLE

1. Derivatives. 2. Primitives and integrals. 3. Elementary functions. 4. Differential equations. 5. Local study of functions. 6. Generalized Taylor expansions. The Euler-Maclaurin summation formula. 7. The gamma function. Dictionary.

V. TOPOLOGICAL VECTOR SPACES

1. Topological vector spaces over a valued field. 2. Convex sets and locally convex spaces. 3. Spaces of continuous linear mappings. 4. Duality in topological vector spaces. 5. Hilbert spaces: elementary theory. Dictionary.

VI. INTEGRATION

1. Convexity inequalities. 2. Riesz spaces. 3. Measures on locally compact spaces. 4. Extension of a measure. L^p spaces. 5. Integration of measures. 6. Vectorial integration. 7. Haar measure. 8. Convolution and representation.

CONTENTS OF THE ELEMENTS OF MATHEMATICS SERIES

LIE GROUPS AND LIE ALGEBRAS

1. Lie algebras.
2. Free Lie algebras.
3. Lie groups.
4. Coxeter groups and Tits systems.
5. Groups generated by reflections.
6. Root systems.

COMMUTATIVE ALGEBRA

1. Flat modules.
2. Localization.
3. Graduations, filtrations, and topologies.
4. Associated prime ideals and primary decomposition.
5. Integers.
6. Valuations.
7. Divisors.

SPECTRAL THEORY

1. Normed algebras.
2. Locally compact groups.

DIFFERENTIAL AND ANALYTIC MANIFOLDS

Summary of results.

CONTENTS

TO THE READER	v
CONTENTS OF THE ELEMENTS OF MATHEMATICS SERIES	ix
INTRODUCTION	xxi
CHAPTER I. ALGEBRAIC STRUCTURES	1
§ 1. Laws of composition; associativity; commutativity	1
1. Laws of composition	1
2. Composition of an ordered sequence of elements	3
3. Associative laws	4
4. Stable subsets. Induced laws	6
5. Permutable elements. Commutative laws	7
6. Quotient laws	11
§ 2. Identity element; cancellable elements; invertible elements ..	12
1. Identity element	12
2. Cancellable elements	14
3. Invertible elements	15
4. Monoid of fractions of a commutative monoid	17
5. Applications: I. Rational integers	20
6. Applications: II. Multiplication of rational integers	22
7. Applications: III. Generalized powers	23
8. Notation	23
§ 3. Actions	24
1. Actions	24
2. Subsets stable under an action. Induced action	26
3. Quotient action	26
4. Distributivity	27
5. Distributivity of one internal law with respect to another.	29
	xi

CONTENTS

§ 4. Groups and groups with operators	30
1. Groups	30
2. Groups with operators	31
3. Subgroups	32
4. Quotient groups	34
5. Decomposition of a homomorphism	37
6. Subgroups of a quotient group	38
7. The Jordan-Hölder theorem	41
8. Products and fibre products	45
9. Restricted sums	47
10. Monogenous groups	48
§ 5. Groups operating on a set	52
1. Monoid operating on a set	52
2. Stabilizer, fixer	54
3. Inner automorphisms	55
4. Orbits	56
5. Homogeneous sets	58
6. Homogeneous principal sets	60
7. Permutation groups of a finite set	61
§ 6. Extensions, solvable groups, nilpotent groups	65
1. Extensions	65
2. Commutators	68
3. Lower central series, nilpotent groups	71
4. Derived series, solvable groups	74
5. p -groups	76
6. Sylow subgroups	78
7. Finite nilpotent groups	80
§ 7. Free monoids, free groups	81
1. Free magmas	81
2. Free monoids	82
3. Amalgamated sum of monoids	84
4. Application to free monoids	88
5. Free groups	89
6. Presentations of a group	90
7. Free commutative groups and monoids	92
8. Exponential notation	94
9. Relations between the various free objects	95
§ 8. Rings	96
1. Rings	96
2. Consequences of distributivity	98

CONTENTS

3. Examples of rings	101
4. Ring homomorphisms	102
5. Subrings	103
6. Ideals	103
7. Quotient rings	105
8. Subrings and ideals in a quotient ring	106
9. Multiplication of ideals	107
10. Product of rings	108
11. Direct decomposition of a ring	110
12. Rings of fractions	112
§ 9. Fields	114
1. Fields	114
2. Integral domains	116
3. Prime ideals	116
4. The field of rational numbers	117
§ 10. Inverse and direct limits	118
1. Inverse systems of magmas	118
2. Inverse limits of actions	119
3. Direct systems of magmas	120
4. Direct limit of actions	123
Exercises for § 1	124
Exercises for § 2	126
Exercises for § 3	129
Exercises for § 4	132
Exercises for § 5	140
Exercises for § 6	147
Exercises for § 7	159
Exercises for § 8	171
Exercises for § 9	174
Exercises for § 10	179
Historical note	180
CHAPTER II. LINEAR ALGEBRA	191
§ 1. Modules	191
1. Modules; vector spaces; linear combinations	191
2. Linear mappings	194
3. Submodules; quotient modules	196

CONTENTS

4. Exact sequences	197
5. Products of modules	200
6. Direct sum of modules	202
7. Intersection and sum of submodules	205
8. Direct sums of submodules	208
9. Supplementary submodules	210
10. Modules of finite length	212
11. Free families. Bases	214
12. Annihilators. Faithful modules. Monogenous modules ...	219
13. Change of the ring of scalars	221
14. Multimodules	224
§ 2. Modules of linear mappings. Duality	227
1. Properties of $\text{Hom}_A(E, F)$ relative to exact sequences ...	227
2. Projective modules	231
3. Linear forms; dual of a module	232
4. Orthogonality	234
5. Transpose of a linear mapping	234
6. Dual of a quotient module. Dual of a direct sum. Dual bases	236
7. Bidual	239
8. Linear equations	240
§ 3. Tensor products	243
1. Tensor product of two modules	243
2. Tensor product of two linear mappings	245
3. Change of ring	246
4. Operators on a tensor product; tensor products as multi-	
modules	247
5. Tensor product of two modules over a commutative ring	249
6. Properties of $E \otimes_A F$ relative to exact sequences	251
7. Tensor products of products and direct sums	254
8. Associativity of the tensor product	258
9. Tensor product of families of multimodules	259
§ 4. Relations between tensor products and homomorphism modules	267
1. The isomorphisms	
$\text{Hom}_B(E \otimes_A F, G) \rightarrow \text{Hom}_A(F, \text{Hom}_B(E, G))$	
and	
$\text{Hom}_C(E \otimes_A F, G) \rightarrow \text{Hom}_A(E, \text{Hom}_C(F, G))$	267
2. The homomorphism $E^* \otimes_A F \rightarrow \text{Hom}_A(E, F)$	268
3. Trace of an endomorphism	273
4. The homomorphism	
$\text{Hom}_C(E_1, F_1) \otimes_C \text{Hom}_C(E_2, F_2)$	
$\rightarrow \text{Hom}_C(E_1 \otimes_C E_2, F_1 \otimes_C F_2)$	274

CONTENTS

§ 5. Extension of the ring of scalars	277
1. Extension of the ring of scalars of a module	277
2. Relations between restriction and extension of the ring of scalars	280
3. Extension of the ring of operators of a homomorphism module	282
4. Dual of a module obtained by extension of scalars	283
5. A criterion for finiteness	284
§ 6. Inverse and direct limits of modules	284
1. Inverse limits of modules	284
2. Direct limits of modules	286
3. Tensor product of direct limits	289
§ 7. Vector spaces	292
1. Bases of a vector space	292
2. Dimension of vector spaces	293
3. Dimension and codimension of a subspace of a vector space	295
4. Rank of a linear mapping	298
5. Dual of a vector space	299
6. Linear equations in vector spaces	304
7. Tensor product of vector spaces	306
8. Rank of an element of a tensor product	309
9. Extension of scalars for a vector space	310
10. Modules over integral domains	312
§ 8. Restriction of the field of scalars in vector spaces	317
1. Definition of K' -structures	317
2. Rationality for a subspace	318
3. Rationality for a linear mapping	319
4. Rational linear forms	320
5. Application to linear systems	321
6. Smallest field of rationality	322
7. Criteria for rationality	323
§ 9. Affine spaces and projective spaces	325
1. Definition of affine spaces	325
2. Barycentric calculus	326
3. Affine linear varieties	327
4. Affine linear mappings	329
5. Definition of projective spaces	331
6. Homogeneous coordinates	331
7. Projective linear varieties	332
8. Projective completion of an affine space	333
9. Extension of rational functions	334

CONTENTS

10. Projective linear mappings.....	335
11. Projective space structure.....	337
§ 10. Matrices	338
1. Definition of matrices	338
2. Matrices over a commutative group.....	339
3. Matrices over a ring	341
4. Matrices and linear mappings	342
5. Block products.....	346
6. Matrix of a semi-linear mapping	347
7. Square matrices.....	349
8. Change of bases	352
9. Equivalent matrices; similar matrices	354
10. Tensor product of matrices over a commutative ring	356
11. Trace of a matrix	358
12. Matrices over a field.....	359
13. Equivalence of matrices over a field	360
§ 11. Graded modules and rings	363
1. Graded commutative groups	363
2. Graded rings and modules	364
3. Graded submodules	367
4. Case of an ordered group of degrees	371
5. Graded tensor product of graded modules	374
6. Graded modules of graded homomorphisms	375
Appendix. Pseudomodules	378
1. Adjunction of a unit element to a pseudo-ring	378
2. Pseudomodules	378
Exercises for § 1	380
Exercises for § 2	386
Exercises for § 3	395
Exercises for § 4	396
Exercises for § 5	398
Exercises for § 6	399
Exercises for § 7	400
Exercises for § 8	409
Exercises for § 9	410
Exercises for § 10	417

CONTENTS

Exercises for § 11	424
Exercise for the Appendix	425
CHAPTER III. TENSOR ALGEBRAS, EXTERIOR ALGEBRAS, SYMMETRIC	
ALGEBRAS.	427
§ 1. Algebras	428
1. Definition of an algebra	428
2. Subalgebras. Ideals. Quotient algebras	429
3. Diagrams expressing associativity and commutativity	431
4. Products of algebras	432
5. Restriction and extension of scalars	433
6. Inverse and direct limits of algebras	434
7. Bases of an algebra. Multiplication table.....	436
§ 2. Examples of algebras	438
1. Endomorphism algebras	438
2. Matrix elements	438
3. Quadratic algebras	439
4. Cayley algebras	441
5. Construction of Cayley algebras. Quaternions	443
6. Algebra of a magma, a monoid, a group.....	446
7. Free algebras	448
8. Definition of an algebra by generators and relations	450
9. Polynomial algebras	452
10. Total algebra of a monoid	454
11. Formal power series over a commutative ring	455
§ 3. Graded algebras	457
1. Graded algebras	457
2. Graded subalgebras, graded ideals of a graded algebra ..	459
3. Direct limits of graded algebras	460
§ 4. Tensor products of algebras	460
1. Tensor product of a finite family of algebras	460
2. Universal characterization of tensor products of algebras	463
3. Modules and multimodules over tensor products of algebras	465
4. Tensor product of algebras over a field	468
5. Tensor product of an infinite family of algebras	470
6. Commutation lemmas	472
7. Tensor product of graded algebras relative to commutation factors	474
8. Tensor product of graded algebras of the same types	480
9. Anticommutative algebras and alternating algebras	482

CONTENTS

§ 5. Tensor algebra. Tensors	484
1. Definition of the tensor algebra of a module	484
2. Functorial properties of the tensor algebra	485
3. Extension of the ring of scalars	489
4. Direct limit of tensor algebras	490
5. Tensor algebra of a direct sum. Tensor algebra of a free module. Tensor algebra of a graded module	491
6. Tensors and tensor notation	492
§ 6. Symmetric algebras	497
1. Symmetric algebra of a module	497
2. Functorial properties of the symmetric algebra	498
3. n -th symmetric power of a module and symmetric multi- linear mappings	500
4. Extension of the ring of scalars	502
5. Direct limit of symmetric algebras	503
6. Symmetric algebra of a direct sum. Symmetric algebra of a free module. Symmetric algebra of a graded module ..	503
§ 7. Exterior algebras	507
1. Definition of the exterior algebra of a module	507
2. Functorial properties of the exterior algebra	508
3. Anticommutativity of the exterior algebra	510
4. n -th exterior power of a module and alternating multi- linear mappings	511
5. Extension of the ring of scalars	513
6. Direct limits of exterior algebras	514
7. Exterior algebra of a direct sum. Exterior algebra of a graded module	515
8. Exterior algebra of a free module	517
9. Criteria for linear independence	519
§ 8. Determinants	522
1. Determinants of an endomorphism	522
2. Characterization of automorphisms of a finite-dimensional free module	523
3. Determinant of a square matrix	524
4. Calculation of a determinant	526
5. Minors of a matrix	528
6. Expansions of a determinant	530
7. Application to linear equations	534
8. Case of a commutative field	536
9. The unimodular group $SL(n, A)$	537

CONTENTS

10. The $A[X]$ -module associated with an A -module endomorphism	538
11. Characteristic polynomial of an endomorphism	540
§ 9. Norms and traces	541
1. Norms and traces relative to a module	541
2. Properties of norms and traces relative to a module	542
3. Norm and trace in an algebra	543
4. Properties of norms and traces in an algebra	545
5. Discriminant of an algebra	549
§ 10. Derivations	550
1. Commutation factors	550
2. General definition of derivations	551
3. Examples of derivations	553
4. Composition of derivations	554
5. Derivations of an algebra A into an A -module	557
6. Derivations of an algebra	559
7. Functorial properties	560
8. Relations between derivations and algebra homomorphisms	561
9. Extension of derivations	562
10. Universal problem for derivations: non-commutative case	567
11. Universal problem for derivations: commutative case	568
12. Functorial properties of K -differentials	570
§ 11. Cogebras, products of multilinear forms, inner products and duality	574
1. Cogebras	574
2. Coassociativity, cocommutativity, counit	578
3. Properties of graded cogebras of type N	584
4. Bigebras and skew-bigebras	585
5. The graded duals $T(M)^{**r}$, $S(M)^{**r}$ and $\bigwedge(M)^{**r}$	587
6. Inner products: case of algebras	594
7. Inner products: case of cogebras	597
8. Inner products: case of bigebras	600
9. Inner products between $T(M)$ and $T(M^*)$, $S(M)$ and $S(M^*)$, $\bigwedge(M)$ and $\bigwedge(M^*)$	603
10. Explicit form of inner products in the case of a finitely generated free module	605
11. Isomorphisms between $\bigwedge^p(M)$ and $\bigwedge^{n-p}(M^*)$ for an n -dimensional free module M	607
12. Application to the subspace associated with a p -vector	608
13. Pure p -vectors. Grassmannians	609

CONTENTS

Appendix. Alternative algebras. Octonions	611
1. Alternative algebras	611
2. Alternative Cayley algebras	613
3. Octonions	615
Exercises for § 1	618
Exercises for § 2	618
Exercises for § 3	626
Exercises for § 4	626
Exercises for § 5	627
Exercises for § 6	632
Exercises for § 7	633
Exercises for § 8	637
Exercises for § 9	644
Exercises for § 10	645
Exercises for § 11	647
Exercises for the Appendix	654
HISTORICAL NOTE ON CHAPTERS II AND III	655
INDEX OF NOTATION	669
INDEX OF TERMINOLOGY	677