GLOBAL EDITION

College Algebra and Trigonometry

SEVENTH EDITION

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3.2 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. In arithmetic, the result of the division

$$\begin{array}{r} 3 \\ 5\overline{)19} \\ \underline{15} \\ 4 \end{array}$$

can be written

$$19 = 5 \cdot \underline{\qquad} + \underline{\qquad}.$$

2. In algebra, the result of the division

$$\begin{array}{r} x + 3 \\ x - 1\overline{)x^2 + 2x + 3} \\ \underline{x^2 - x\phantom{{}+3}} \\ 3x + 3 \\ \underline{3x - 3} \\ 6 \end{array}$$

can be written $x^2 + 2x + 3 =$ $(x - 1)(\underline{\qquad}) + \underline{\qquad}$.

3. To perform the division in **Exercise 2** using synthetic division, we begin by writing the following.

$$\underline{\qquad}\overline{)\ \underline{\qquad} \quad 2 \quad 3}$$

4. To perform the division

$$x + 2\overline{)x^3 + 4x + 2}$$

using synthetic division, we begin by writing the following.

$$\underline{\qquad}\overline{)1 \quad \underline{\qquad} \quad 4 \quad 2}$$

5. To perform the division

$$x - 3\overline{)x^3 + 6x^2 + 2x}$$

using synthetic division, we begin by writing the following.

$$\underline{\qquad}\overline{)1 \quad \underline{\qquad} \quad 2 \quad \underline{\qquad}}$$

6. Consider the following function.

$$f(x) = 2x^4 + 6x^3 - 5x^2 + 3x + 8$$

$$f(x) = (x - 2)(2x^3 + 10x^2 + 15x + 33) + 74$$

By inspection, we can state that $f(2) = \underline{\qquad}$.

Use synthetic division to perform each division. ***See Example 1.***

7. $\dfrac{x^3 + 3x^2 + 11x + 9}{x + 1}$

8. $\dfrac{x^3 + 7x^2 + 13x + 6}{x + 2}$

9. $\dfrac{5x^4 + 5x^3 + 2x^2 - x - 3}{x + 1}$

10. $\dfrac{2x^4 - x^3 - 7x^2 + 7x - 10}{x - 2}$

11. $\dfrac{x^4 + 2x^3 + 2x^2 + 5x + 2}{x + 2}$

12. $\dfrac{x^4 + 5x^3 + 4x^2 - 3x + 9}{x + 3}$

13. $\dfrac{x^5 + 3x^4 + 2x^3 + 2x^2 + 3x + 1}{x + 2}$

14. $\dfrac{x^6 - 3x^4 + 2x^3 - 6x^2 - 5x + 3}{x + 2}$

15. $\dfrac{-9x^3 + 8x^2 - 7x + 2}{x - 2}$

16. $\dfrac{-11x^4 + 2x^3 - 8x^2 - 4}{x + 1}$

17. $\dfrac{\frac{1}{5}x^3 - \frac{8}{25}x^2 + \frac{8}{125}x - \frac{1}{625}}{x - \frac{1}{5}}$

18. $\dfrac{x^3 + x^2 + \frac{1}{2}x + \frac{1}{8}}{x + \frac{1}{2}}$

19. $\dfrac{x^4 - 3x^3 - 4x^2 + 12x}{x - 2}$ **20.** $\dfrac{x^4 - x^3 - 5x^2 - 3x}{x + 1}$

21. $\dfrac{x^3 - 512}{x - 8}$ **22.** $\dfrac{x^4 - 1}{x - 1}$ **23.** $\dfrac{x^5 + 32}{x + 2}$ **24.** $\dfrac{x^7 + 1}{x + 1}$

Use synthetic division to divide $f(x)$ by $x - k$ for the given value of k. Then express $f(x)$ in the form $f(x) = (x - k)q(x) + r$.

25. $f(x) = 2x^3 + x^2 + x - 8; \quad k = -1$

26. $f(x) = 4x^3 + x^2 + x - 7; \quad k = -1$

27. $f(x) = x^3 + 5x^2 + 9x + 6; \quad k = -2$

28. $f(x) = -x^3 + x^2 + 3x - 2; \quad k = 2$

29. $f(x) = 4x^4 - 2x^3 - 19x^2 - x; \quad k = 4$

30. $f(x) = 2x^4 + x^3 - 15x^2 + 3x; \quad k = -3$

31. $f(x) = 3x^4 + 8x^3 - 11x^2 + 64; \quad k = -2$

32. $f(x) = -5x^4 + x^3 + 2x^2 + 3x + 1; \quad k = 1$

For each polynomial function, use the remainder theorem to find $f(k)$. ***See Example 2.***

33. $f(x) = x^2 + 5x + 6; \quad k = -2$ **34.** $f(x) = x^2 - 4x + 2; \quad k = -1$

35. $f(x) = 2x^2 - 3x - 3; \quad k = 2$ **36.** $f(x) = -x^3 + 8x^2 + 63; \quad k = 4$

37. $f(x) = -2x^3 - 14x^2 - 13x - 5; \quad k = -6$ **38.** $f(x) = 2x^3 - 3x^2 - 5x + 4; \quad k = 2$

39. $f(x) = x^2 - 9x + 2; \quad k = 4 + i$ **40.** $f(x) = x^2 - x + 3; \quad k = 3 - 2i$

41. $f(x) = x^2 + 4; \quad k = 2i$ **42.** $f(x) = 2x^2 + 10; \quad k = i\sqrt{5}$

43. $f(x) = 2x^2 + 49; \quad k = 6i$

44. $f(x) = x^4 + 6x^3 + 9x^2 + 3x - 3; \quad k = 4$

45. $f(x) = 6x^4 + x^3 - 8x^2 + 5x + 6; \quad k = \frac{1}{2}$

46. $f(x) = 6x^3 - 31x^2 - 15x; \quad k = -\frac{1}{2}$

Use synthetic division to determine whether the given number k is a zero of the polynomial function. If it is not, give the value of $f(k)$. ***See Examples 2 and 3.***

47. $f(x) = x^2 - 7x + 12; \quad k = 3$ **48.** $f(x) = x^2 + 4x - 5; \quad k = -5$

49. $f(x) = x^3 - 3x^2 + 4x - 4; \quad k = 2$ **50.** $f(x) = x^3 + 2x^2 - x + 6; \quad k = -3$

51. $f(x) = 5x^3 - 8x^2 - 21x + 15; \quad k = 1$ **52.** $f(x) = 2x^3 + 9x^2 - 16x + 12; \quad k = 1$

53. $f(x) = x^3 + 7x^2 + 10x; \quad k = 0$ **54.** $f(x) = 2x^3 - 3x^2 - 5x; \quad k = 0$

55. $f(x) = 7x^4 + 3x^3 - x + 5; \quad k = \frac{2}{7}$ **56.** $f(x) = 16x^4 + 3x^2 - 2; \quad k = \frac{1}{2}$

57. $f(x) = x^2 - 2x + 2; \quad k = 1 - i$ **58.** $f(x) = x^2 - 4x + 5; \quad k = 2 - i$

59. $f(x) = x^2 + 3x + 4; \quad k = 2 + i$ **60.** $f(x) = x^2 - 3x + 5; \quad k = 1 - 2i$

61. $f(x) = 4x^4 + x^2 + 17x + 3; \quad k = -\frac{3}{2}$

62. $f(x) = 3x^4 + 13x^3 - 10x + 8; \quad k = -\frac{4}{3}$

63. $f(x) = x^3 + 2x^2 - 2x + 2; \quad k = 2 + i$

64. $f(x) = 2x^3 - x^2 + 3x - 5; \quad k = 2 - i$

Relating Concepts

For individual or collaborative investigation *(Exercises 65–74)*

The remainder theorem indicates that when a polynomial $f(x)$ is divided by $x - k$, the remainder is equal to $f(k)$. Consider the polynomial function

$$f(x) = x^3 - 2x^2 - x + 2.$$

Use the remainder theorem to find each of the following. Then determine the coordinates of the corresponding point on the graph of $f(x)$.

65. $f(-2)$ **66.** $f(-1)$ **67.** $f\left(-\frac{1}{2}\right)$ **68.** $f(0)$

69. $f(1)$ **70.** $f\left(\frac{3}{2}\right)$ **71.** $f(2)$ **72.** $f(3)$

73. Use the results from **Exercises 65–72** to plot eight points on the graph of $f(x)$. Join these points with a smooth curve.

74. Apply the method above to graph $f(x) = -x^3 - x^2 + 2x$. Use x-values $-3, -1, \frac{1}{2}$, and 2 and the fact that $f(0) = 0$.

3.3 Zeros of Polynomial Functions

- Factor Theorem
- Rational Zeros Theorem
- Number of Zeros
- Conjugate Zeros Theorem
- Zeros of a Polynomial Function
- Descartes' Rule of Signs

Factor Theorem

Consider the polynomial function

$$f(x) = x^2 + x - 2,$$

which is written in factored form as

$$f(x) = (x - 1)(x + 2).$$

For this function, $f(1) = 0$ and $f(-2) = 0$, and thus 1 and -2 are zeros of $f(x)$. Notice the special relationship between each linear factor and its corresponding zero. The **factor theorem** summarizes this relationship.

Factor Theorem

For any polynomial function $f(x)$, $x - k$ is a factor of the polynomial if and only if $f(k) = 0$.

EXAMPLE 1 Determining Whether *x* − *k* Is a Factor

Determine whether $x - 1$ is a factor of each polynomial.

(a) $f(x) = 2x^4 + 3x^2 - 5x + 7$

(b) $f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1$

SOLUTION

(a) By the factor theorem, $x - 1$ will be a factor if $f(1) = 0$. Use synthetic division and the remainder theorem to decide.

$$\begin{array}{r|rrrrr} 1 & 2 & 0 & 3 & -5 & 7 \\ & & 2 & 2 & 5 & 0 \\ \hline & 2 & 2 & 5 & 0 & 7 \end{array} \longleftarrow f(1) = 7$$

Use a zero coefficient for the missing term.

The remainder is 7, not 0, so $x - 1$ is not a factor of $2x^4 + 3x^2 - 5x + 7$.

(b)
$$\begin{array}{r|rrrrrrl} 1 & 3 & -2 & 1 & -8 & 5 & 1 & \longleftarrow f(x) = 3x^5 - 2x^4 + x^3 - 8x^2 + 5x + 1 \\ & & 3 & 1 & 2 & -6 & -1 & \\ \hline & 3 & 1 & 2 & -6 & -1 & 0 & \longleftarrow f(1) = 0 \end{array}$$

Because the remainder is 0, $x - 1$ is a factor. Additionally, we can determine from the coefficients in the bottom row that the other factor is

$$3x^4 + 1x^3 + 2x^2 - 6x - 1.$$

Thus, we can express the polynomial in factored form.

$$f(x) = (x - 1)(3x^4 + x^3 + 2x^2 - 6x - 1)$$

✓ **Now Try Exercises 9 and 11.**

We can use the factor theorem to factor a polynomial of greater degree into linear factors of the form $ax - b$.

EXAMPLE 2 Factoring a Polynomial Given a Zero

Factor $f(x) = 6x^3 + 19x^2 + 2x - 3$ into linear factors given that -3 is a zero.

SOLUTION Because -3 is a zero of f, $x - (-3) = x + 3$ is a factor.

$$\begin{array}{r|rrrr} -3 & 6 & 19 & 2 & -3 \\ & & -18 & -3 & 3 \\ \hline & 6 & 1 & -1 & 0 \end{array} \qquad \text{Use synthetic division to divide } f(x) \text{ by } x + 3.$$

The quotient is $6x^2 + x - 1$, which is the factor that accompanies $x + 3$.

$$f(x) = (x + 3)(6x^2 + x - 1)$$

$$f(x) = \underbrace{(x + 3)(2x + 1)(3x - 1)}_{\text{These factors are all linear.}} \qquad \text{Factor } 6x^2 + x - 1.$$

✓ **Now Try Exercise 21.**

Rational Zeros Theorem

The **rational zeros theorem** gives a method to determine all possible candidates for rational zeros of a polynomial function with integer coefficients.

Rational Zeros Theorem

If $\frac{p}{q}$ is a rational number written in lowest terms, and if $\frac{p}{q}$ is a zero of f, a polynomial function with integer coefficients, then p is a factor of the constant term and q is a factor of the leading coefficient.

Proof $f\left(\frac{p}{q}\right) = 0$ because $\frac{p}{q}$ is a zero of $f(x)$.

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \qquad \text{Definition of zero of } f$$

$$a_n\left(\frac{p^n}{q^n}\right) + a_{n-1}\left(\frac{p^{n-1}}{q^{n-1}}\right) + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0 \qquad \text{Power rule for exponents}$$

$$a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} = -a_0q^n \qquad \text{Multiply by } q^n\text{. Subtract } a_0q^n\text{.}$$

$$p(a_np^{n-1} + a_{n-1}p^{n-2}q + \cdots + a_1q^{n-1}) = -a_0q^n \qquad \text{Factor out } p\text{.}$$

LOOKING AHEAD TO CALCULUS

Finding the derivative of a polynomial function is one of the basic skills required in a first calculus course. For the functions

$$f(x) = x^4 - x^2 + 5x - 4,$$
$$g(x) = -x^6 + x^2 - 3x + 4,$$
and $$h(x) = 3x^3 - x^2 + 2x - 4,$$

the derivatives are

$$f'(x) = 4x^3 - 2x + 5,$$
$$g'(x) = -6x^5 + 2x - 3,$$
and $$h'(x) = 9x^2 - 2x + 2.$$

Notice the use of the “prime” notation. For example, the derivative of $f(x)$ is denoted $f'(x)$.

Look for the pattern among the exponents and the coefficients. Using this pattern, what is the derivative of

$$F(x) = 4x^4 - 3x^3 + 6x - 4?$$

The answer is at the top of the next page.

Answer to Looking Ahead to Calculus:

$F'(x) = 16x^3 - 9x^2 + 6$

This result shows that $-a_0q^n$ equals the product of the two factors p and $(a_np^{n-1} + \cdots + a_1q^{n-1})$. For this reason, p must be a factor of $-a_0q^n$. Because it was assumed that $\frac{p}{q}$ is written in lowest terms, p and q have no common factor other than 1, so p is not a factor of q^n. Thus, p must be a factor of a_0. In a similar way, it can be shown that q is a factor of a_n.

EXAMPLE 3 Using the Rational Zeros Theorem

Consider the polynomial function.

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

(a) List all possible rational zeros.

(b) Find all rational zeros and factor $f(x)$ into linear factors.

SOLUTION

(a) For a rational number $\frac{p}{q}$ to be a zero, p must be a factor of $a_0 = 2$, and q must be a factor of $a_4 = 6$. Thus, p can be ± 1 or ± 2, and q can be ± 1, ± 2, ± 3, or ± 6. The possible rational zeros $\frac{p}{q}$ are ± 1, ± 2, $\pm \frac{1}{2}$, $\pm \frac{1}{3}$, $\pm \frac{1}{6}$, and $\pm \frac{2}{3}$.

(b) Use the remainder theorem to show that 1 is a zero.

Use "trial and error" to find zeros.

$$\begin{array}{r|rrrrr} 1 & 6 & 7 & -12 & -3 & 2 \\ & & 6 & 13 & 1 & -2 \\ \hline & 6 & 13 & 1 & -2 & 0 \end{array} \longleftarrow f(1) = 0$$

The 0 remainder shows that 1 is a zero. The quotient is $6x^3 + 13x^2 + x - 2$.

$$f(x) = (x - 1)(6x^3 + 13x^2 + x - 2) \quad \text{Begin factoring } f(x).$$

Now, use the quotient polynomial and synthetic division to find that -2 is a zero.

$$\begin{array}{r|rrrr} -2 & 6 & 13 & 1 & -2 \\ & & -12 & -2 & 2 \\ \hline & 6 & 1 & -1 & 0 \end{array} \longleftarrow f(-2) = 0$$

The new quotient polynomial is $6x^2 + x - 1$. Therefore, $f(x)$ can now be completely factored as follows.

$$f(x) = (x - 1)(x + 2)(6x^2 + x - 1)$$

$$f(x) = (x - 1)(x + 2)(3x - 1)(2x + 1)$$

Setting $3x - 1 = 0$ and $2x + 1 = 0$ yields the zeros $\frac{1}{3}$ and $-\frac{1}{2}$. In summary, the rational zeros are 1, -2, $\frac{1}{3}$, and $-\frac{1}{2}$. These zeros correspond to the x-intercepts of the graph of $f(x)$ in **Figure 18.** The linear factorization of $f(x)$ is as follows.

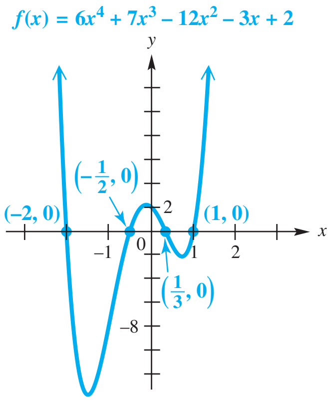

Figure 18

$$f(x) = 6x^4 + 7x^3 - 12x^2 - 3x + 2$$

$$f(x) = (x - 1)(x + 2)(3x - 1)(2x + 1)$$

Check by multiplying these factors.

✔ **Now Try Exercise 39.**

NOTE Once we obtained the quadratic factor

$$6x^2 + x - 1$$

in **Example 3,** we were able to complete the work by factoring it directly. Had it not been easily factorable, we could have used the quadratic formula to find the other two zeros (and factors).

CAUTION ***The rational zeros theorem gives only possible rational zeros. It does not tell us whether these rational numbers are actual zeros.*** We must rely on other methods to determine whether or not they are indeed zeros. Furthermore, the polynomial must have integer coefficients.

To apply the rational zeros theorem to a polynomial with fractional coefficients, multiply through by the least common denominator of all the fractions. For example, any rational zeros of $p(x)$ defined below will also be rational zeros of $q(x)$.

$$p(x) = x^4 - \frac{1}{6}x^3 + \frac{2}{3}x^2 - \frac{1}{6}x - \frac{1}{3}$$

$$q(x) = 6x^4 - x^3 + 4x^2 - x - 2 \qquad \text{Multiply the terms of } p(x) \text{ by 6.}$$

Number of Zeros The **fundamental theorem of algebra** says that every function defined by a polynomial of degree 1 or more has a zero, which means that every such polynomial can be factored.

Carl Friedrich Gauss (1777–1855)

The **fundamental theorem of algebra** was first proved by Carl Friedrich Gauss in his doctoral thesis in 1799, when he was 22 years old.

Fundamental Theorem of Algebra

Every function defined by a polynomial of degree 1 or more has at least one complex zero.

From the fundamental theorem, if $f(x)$ is of degree 1 or more, then there is some number k_1 such that $f(k_1) = 0$. By the factor theorem,

$$f(x) = (x - k_1)q_1(x), \quad \text{for some polynomial } q_1(x).$$

If $q_1(x)$ is of degree 1 or more, the fundamental theorem and the factor theorem can be used to factor $q_1(x)$ in the same way. There is some number k_2 such that $q_1(k_2) = 0$, so

$$q_1(x) = (x - k_2)q_2(x)$$

and

$$f(x) = (x - k_1)(x - k_2)q_2(x).$$

Assuming that $f(x)$ has degree n and repeating this process n times gives

$$f(x) = a(x - k_1)(x - k_2) \cdots (x - k_n). \qquad a \text{ is the leading coefficient.}$$

Each of these factors leads to a zero of $f(x)$, so $f(x)$ has the n zeros $k_1, k_2, k_3, \ldots, k_n$. This result suggests the **number of zeros theorem.**

Number of Zeros Theorem

A function defined by a polynomial of degree n has *at most* n distinct zeros.