1

The Asymptotic Perturbation Method for Nonlinear Oscillators

1.1 Introduction

Oscillations are a fundamental topic in physics. When a system is near its equilibrium point, it begins to oscillate, but if the displacement increases, then the nonlinear terms are not negligible. The starting point is the differential equation for the harmonic oscillator

$$\frac{d^2X}{dt} + \omega^2 X(t) = 0 \tag{1.1}$$

where $X(t)$ is the displacement and ω the circular frequency. The most general solution is

$$X(t) = 2\rho\cos(-\omega t + \theta) \tag{1.2}$$

where ρ and θ are fixed by the initial conditions (the Cauchy problem)

if $X(0) = X_0$ for the displacement
and $\dot{X}(0) = \dot{X}_0$ for the initial velocity

then we easily get

$$2\rho = \sqrt{\left((X_0)^2 + \left(\frac{\dot{X}_0}{\omega}\right)^2\right)} \tag{1.3}$$

and

$$\tan\theta = \left(\frac{\dot{X}_0}{\omega X_0}\right) \tag{1.4}$$

Now, we can consider a weakly nonlinear part in the differential Eq. (1.1) or, on the contrary, a strongly nonlinear part but with small solutions. The first consequence is that the amplitude and the phase are slowly varying with time, so we can introduce another slow time

$$\tau = \varepsilon^q t \tag{1.5}$$

where ε is a bookkeeping device and q is a rational number that will be chosen afterwards. If we want to study the asymptotic solution behavior ($t \to \infty$) and $\varepsilon \to 0$, then

Asymptotic Perturbation Methods: For Nonlinear Differential Equations in Physics,
First Edition. Attilio Maccari.

τ must assume finite values. So, we assume that an approximate solution is given by

$$X(t) = 2\rho(\tau)\cos(-\omega t + \theta(\tau)) = (\rho(\tau)\exp(-i\omega t + i\theta) + c.c.) \tag{1.6}$$

or better

$$X(t) = \varepsilon^{(1+r)}\Psi_0 + (\varepsilon\Psi_1 \exp(-i\omega t) + \varepsilon^2\Psi_2 \exp(-2\omega t) + \varepsilon^3\Psi_3 \exp(-3i\omega t) + c.c. + h.o.t.) \tag{1.7}$$

where $c.\,c.$ stands for complex conjugate and $h.\,o.\,t.$ for higher order terms.

Following this path, we are mixing the most important features of two well-known perturbation methods, the harmonic balance and the multiple scale methods (for more details about these two perturbation methods, see Refs. [202, 203, 249]).

If we consider a weakly nonlinear differential equation

$$\frac{d^2X}{dt} + \omega^2 X(t) = NL \tag{1.8}$$

where NL stands for the nonlinear part, for instance,

$$aX(t)^2 + bX(t)^3 \tag{1.9}$$

we can insert the solution (1.7) in the nonlinear Eq. (1.8) and with some algebra manipulation, we get for $n = 0$

$$\omega^2\varepsilon^{(1+r)}\Psi_0 = 2a\varepsilon^2|\Psi|^2 \tag{1.10}$$

then $r = 1$, for $n = 2$

$$-3\omega^2\varepsilon^2\Psi_2 = a\varepsilon^2\Psi^2 \tag{1.11}$$

and for $n = 1$

$$-2i\omega\varepsilon^q\psi_\tau = 2a(\varepsilon^{82} + r)\Psi_0\Psi + \varepsilon^2\Psi_2(c.c.\Psi) + 3b\varepsilon^2|\Psi|^2\Psi \tag{1.12}$$

then, $q = 2$ for the proper nonlinear term balance and with some algebra manipulation

$$\frac{d\Psi}{d\tau} = \frac{iA}{2\omega}|\Psi|^2\Psi \tag{1.13}$$

where

$$A = \frac{10a^2}{3\omega^2} + b \tag{1.14}$$

$$\frac{d\rho}{d\tau} = 0 \quad \frac{d\theta}{d\tau} = \frac{A}{2\omega}\rho^2 \tag{1.15}$$

We observe that the variable change (1.5) implies that

$$\frac{d}{dt} \to -in\omega + \varepsilon^q \frac{d}{d\tau} \tag{1.16}$$

when the temporal differential operator acts on the function

$$\Psi_n(\tau)\exp(-in\omega t) \tag{1.17}$$

From Eq. (1.10), we can see that the approximate solution is always periodic, the amplitude is constant, but the period changes and becomes

$$T = \frac{2\pi}{\Omega} \quad \text{where} \quad \Omega = \omega - \frac{A}{2\omega}\rho^2 \tag{1.18}$$

However, if

$$b = -\left(\frac{10a^2}{3\omega^2}\right) \tag{1.19}$$

the period does not change and is equal to the linear case period.

In this chapter, we want to extend this method and study a generalized Van der Pol–Duffing oscillator in resonance with a periodic excitation

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) + 2F\cos(\Omega t) \tag{1.20}$$

We use the asymptotic perturbation (AP) method based on Fourier expansion and time rescaling (see above) and demonstrate through a second-order perturbation analysis the existence of one or two limit cycles. Moreover, we identify a sufficient condition to obtain a doubly periodic motion when a second low frequency appears, in addition to the forcing frequency. The comparison with the solution obtained by the numerical integration confirms the validity of our analysis.

1.2 Nonlinear Dynamical Systems

The study of nonlinear dynamical systems has interested many researchers, and various methods have been used. Historically, the AP method was first applied in order to study the most important characteristics of a nonlocal oscillator [112, 113, 118].

We now devote our attention to the following type of nonlinear equation

$$\ddot{X}(t) + f(X(t)) = g(X(t), \dot{X}(t)) \tag{1.21}$$

where the dot denotes differentiation with respect to the time and the functions $f(x)$ and $g(x, y)$ are supposed to be analytic.

The limit cycles of the modified Van der Pol equation

$$\ddot{X}(t) + X(t) + X^3(t) = \varepsilon(1 - X^2(t))\dot{X}(t) \tag{1.22}$$

have been studied in Ref. [23] by means of a time transformation method.

Phase portraits and dynamical properties of the equation

$$\ddot{X}(t) + (\alpha + \beta X^2(t))\dot{X}(t) + \gamma X(t) + \delta X^3(t) = 0 \tag{1.23}$$

have been investigated with the methods of differentiable dynamics [74] and the equation

$$\ddot{X}(t) + X(t) = \varepsilon f(X(t), \dot{X}(t)) \tag{1.24}$$

with the method of averaging, the KBM method, the method of multiple scales, and the Poincaré–Lindstedt method [202, 203].

Note that Eqs. (1.22)–(1.24) belong to the general class (1.21) and are characterized by the fact that $f(x)$ is an odd function of x.

We restrict our study to the following particular case of Eq. (1.21)

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) \tag{1.25}$$

Eq. (1.5) can be considered a generalized Van der Pol–Duffing equation because it includes as particular cases the Van der Pol oscillator ($f_2, f_3, g_1 = 0$ and $g_0 = -g_2 \neq 0$) and the Duffing equation ($f_2 = g_1 = g_2 = 0$ and $g_0 = f_3 \neq 0$). Many authors have studied the problem of approximating the limit cycle of the Van der Pol equation. Stokes [249] used the nonlinear Galerkin method and developed a series representation; Deprit and Schmidt [47] utilized the Poincaré–Lindstedt method to find the amplitude and frequency of the limit cycle; and Garcia-Magallo and Bejarano [57] considered a generalized Van der Pol equation by means of the harmonic balance method. The steady-state behavior of the Van der Pol oscillator has also been studied by integral manifold methods and symbolic manipulation packages by Gilsinn [59, 61]. Mehri and Ghorashi [195] considered the periodically forced Duffing equation in order to establish sufficient conditions to have a periodic solution, and Qaisi [233] studied a similar problem using an analytical approach based on the power series method. In a series of papers [69–71], Hassan used the higher order method of multiple scales with reconstitution and the harmonic balance method to determine the periodic state response of the Duffing oscillator.

In our treatment of Eq. (1.25), no conditions are imposed on the coefficients f_2, f_3, g_1, and g_2, which can be of order 1. Only the dissipative coefficient g_0 is supposed to be of order e^2. Eq. (1.25) transforms into

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = \varepsilon^2 g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) \tag{1.26}$$

In the second section, we calculate the approximate solution good to the order of ε^4 and construct accurate expressions for the limit cycle of Eq. (1.26). Moreover, we demonstrate that, in the first approximation, the behavior of the solution can be described by means of a model system of differential equations, which represents the characteristics of Eq. (1.26) by means of a reduced set of parameters.

Usually, perturbation analysis is carried out only to the first order because, in many cases, a second order-calculation does not change the qualitative behavior of the solution. However, in Section 1.2, we demonstrate that if the parameters are appropriately chosen, we can find two limit cycles and can calculate their positions only by a second-order perturbation analysis.

In Section 1.3, a comparison with the results of the numerical integration permits discussion of the validity of the AP method.

In Section 1.4, we treat an extension of Eq. (1.26) that is a nonlinear oscillator forced by a small periodic excitation, of order e^2, in resonance with the natural frequency of the oscillator

$$\ddot{X}(t) + X(t) + f_2X^2(t) + f_3X^3(t) = \varepsilon^2 g_0\dot{X}(t) + g_1X(t)\dot{X}(t) + g_2X^2(t)\dot{X}(t) + 2\varepsilon^2 f\cos(t) \tag{1.27}$$

We demonstrate that, under appropriate conditions, a stable limit cycle appears and calculate the relative approximate solution. Moreover, we derive sufficient conditions for the existence of a doubly periodic motion when the fundamental oscillation is subjected to a slight modulation, with an amplitude proportional to the magnitude of the periodic excitation.

Finally, in the last section, we briefly recapitulate the most important results and indicate some possible generalizations of the present study.

1.3 The Approximate Solution

The AP method we use to calculate the approximate solution was first developed in Refs. [1, 2], and then in this section, we sketch the main steps of this perturbation technique.

First of all, we now introduce a rational number

$$q = \text{rational number} \tag{1.28}$$

the temporal rescaling

$$t = e^{q} t \tag{1.29}$$

where the rational number q will be fixed afterwards because it establishes to what extent we can push the temporal asymptotic limit in such a way that the nonlinear effects become consistent and not negligible. If $t \to \infty$, then $\varepsilon \to 0$, when τ assumes a finite value.

If we take $\varepsilon = 0$ in Eqs. (1.26) and neglect nonlinear terms, we see that it admits simple harmonic solutions $X(t) = A \exp(-\mathrm{it}) + c.c.$, where A is a constant depending on initial conditions and $c.c.$ stands for complex conjugate. Nonlinear effects induce a modulation of the amplitude A and the appearance of higher harmonics. The modulation is best described in terms of the rescaled variable t that accounts for the need to look on larger time scales, to obtain a nonnegligible contribution from the nonlinear term.

The assumed solution $X(t)$ of (1.26) can be expressed by means of a power series in the expansion parameter ε, we formally write

$$X(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\gamma_n} \Psi_n(t, \varepsilon) \exp(-\mathrm{int}) \tag{1.30a}$$

with $\gamma_n = |n|$ for $n \neq 0$, and $\gamma_0 = r$ is a positive number, which will be fixed later on; in consequence of the reality of (1.30a)

$$\Psi_n(t, \varepsilon) = c.c.(\Psi_{(-n)}(t, \varepsilon)) \tag{1.30b}$$

The assumed solution (1.30a) can be considered a combination of the different harmonics, solutions of the linear equation, i.e. of the equation obtained after neglecting all the nonlinear terms, and the coefficients of this combination depend on τ and ε.

Eq. (1.30a) can be written more explicitly

$$X(t) = \varepsilon^{r}\Psi_0(t;\varepsilon) + \varepsilon\Psi_1(t;\varepsilon)\exp(-\mathrm{it}) + \varepsilon^{2}\Psi_2(t;\varepsilon)\exp(-2\mathrm{it}) + \varepsilon^{3}\Psi_3(t;\varepsilon)\exp(-3\mathrm{it})o(\varepsilon^{4}) \tag{1.30c}$$

The functions $\Psi_n(t, \varepsilon)$ depend on the parameter ε, and we suppose that Ψ_n's limit for $\varepsilon \to 0$ exists and is finite and, moreover, they can be expanded in power series of ε, i.e.

$$\Psi_n(\tau; \varepsilon) = \sum_{i=0}^{\infty} \varepsilon^i \Psi_n^{(i)}(\tau) \tag{1.31}$$

In the following, for simplicity, we use the abbreviations $\Psi_n^{(0)} = \Psi_n$ for $n \neq 1$ and $\Psi_1^{(0)} = \Psi$ for $n = 1$.

Note that the variable change (1.29) implies that

$$\frac{(\partial \psi_n \exp(-\text{int}))}{\partial t} = \left(-\text{in}\psi_n + \varepsilon^q \frac{\partial \psi_n}{\partial t} \right) \exp(-\text{int}) \tag{1.32}$$

After inserting this expansion into Eq. (1.26), we obtain equations for every harmonic and for a fixed order of approximation, which are right for the purpose of determining the coefficients.

For $n = 0$, we obtain

$$\varepsilon^r \Psi_0 + 2f_2 |\Psi|^2 \varepsilon^2 + o(\varepsilon^4, \varepsilon^{r+2}) = 0 \tag{1.33a}$$

A correct balance of terms shows $r = 2$, and then we derive the following relation

$$\varepsilon^2 \Psi_0 = -2\varepsilon^2 f_2(|\Psi|^2) + O(\varepsilon^4) \tag{1.33b}$$

For $n = 2$, taking into account Eq. (1.32), we have

$$-3\varepsilon^2 \Psi_2 + f_2 \Psi^2 \varepsilon^2 = -ig_1 \varepsilon^2 \Psi^2 + o(\varepsilon^4, \varepsilon^{2+q}) \tag{1.34a}$$

and then

$$\varepsilon^2 \Psi_2 = \frac{f_2 + ig_1}{3} \varepsilon^2 \Psi^2 + o(\varepsilon^4) \tag{1.34b}$$

For $n = 1$, Eq. (1.26) yields for the right-hand side

$$-2i \frac{d\Psi}{dt} \varepsilon^{1+q} + 2f_2(\Psi_0 \Psi \varepsilon^3 + \Psi_2(c.c.(\Psi))\varepsilon^3) + 3f_3 |\Psi|^2 \Psi \varepsilon^3 \tag{1.35a}$$

and for the left-hand side

$$i\varepsilon^3 g_0 \Psi - ig_1(\Psi_0 \Psi \varepsilon^3 + \Psi \Psi_2(c.c.\Psi)\varepsilon^3) - i\varepsilon^3 g_2 |\Psi|^2 \Psi + o(\varepsilon^5, \varepsilon^{1+2q}) \tag{1.35b}$$

If $q = 2$, the first term has the same magnitude order of nonlinear terms.

Taking into account Eqs. (1.33b) and (1.34b), we can derive a differential equation, which involves only Ψ,

$$\frac{d\Psi}{dt} = \alpha_1 \Psi + (\beta_1 + i\beta_2)|\Psi|^2 \Psi \tag{1.36}$$

with

$$\alpha_1 = \frac{g_0}{2} \tag{1.37}$$

$$\beta_1 = \frac{g_2}{2} - \frac{g_1 f_2}{2} \tag{1.38}$$

$$\beta_2 = \frac{g_1^2}{6} - \frac{3}{2} f_3 + \frac{5}{3} f_2^2 \tag{1.39}$$

Substituting the polar form

$$\Psi(\tau) = \rho(\tau)\exp(i\theta(\tau)) \tag{1.40}$$

into Eq. (1.36), and separating real and imaginary parts, we arrive at the following model system:

$$\frac{d\rho}{dt} = \alpha_1\rho + \beta_1\rho^3 \tag{1.41}$$

$$\frac{dJ}{dt} = \beta_2\rho^2 \tag{1.42}$$

As we can see from Eqs. (1.30c), (1.31), and (1.40), the approximate solution of Eq. (1.26) can be written as a sum of a contribution of order ε and a contribution of order ε^2

$$\begin{aligned} X(t) &= \varepsilon X_1(t) + \varepsilon^2 X_2(t) + o(\varepsilon^3),\\ X_1(t) &= 2\rho(\tau)\cos(-t+\theta(\tau)),\\ X_2(t) &= -2f_2\rho^2(\tau) + \frac{2}{3}f_2\rho^2(\tau)\cos(-2t+2\theta(\tau)) + \frac{2}{3}g_1\rho^2(\tau)\sin(-2t+2\theta(\tau)) \end{aligned} \tag{1.43}$$

By inspection of Eq. (1.41), which can be easily integrated, we conclude that a stable steady-state response is possible if $\alpha_1 > 0$ and $\beta_1 < 0$. In this case, we obtain a stable equilibrium point, which corresponds to a stable limit cycle for Eq. (1.26), and its approximate expression is given by (1.43), with

$$\rho(t) = \rho_E = \sqrt{-\frac{\alpha_1}{\beta_1}} = \text{constant}, \theta(t) = \beta_2\rho_E t \tag{1.44}$$

The natural frequency of the oscillator is subject to a slight modification and becomes

$$\omega_E = \omega - \beta_2\rho_E \tag{1.45}$$

If we want to improve the validity of the approximate solution, we must include higher order terms. However, we can easily conclude that $\Psi_0^{(1)} = \Psi_1^{(1)} = \Psi_2^{(1)} = 0$ (for their definition, see Eq. (1.31)). Indeed, we consider Eq. (1.26) for $n = 0$ and Eqs. (1.33b) and (1.34a) for $n = 0$ and $n = 2$ in such a way to obtain

$$\Psi_0^{(1)} = -2f_2\left(\Psi_1^{(1)}(c.c.\Psi) + \Psi c.c.\left(\Psi_1^{(1)}\right)\right),\quad \Psi_2^{(1)} = \frac{2}{3}(f_2 + ig_1)\Psi_1^{(1)}\Psi \tag{1.46}$$

After inserting (1.26b) into (1.26a), we see that the resulting equation is satisfied if $\Psi_1^{(1)} = 0$. Recall that we can always assume that the initial condition is $\Psi_1^{(1)}(0) = 0$, because the initial conditions associated with equation (1.25), $X(0) = X_0$ and $\dot{X}(0) = \dot{X}_0$, can be used to determine $\Psi(0) = \rho(0)\exp(i\theta(0))$.

A valid higher order approximation can be derived only if we take into account $\Psi_1^{(2)}, \Psi_2^{(2)}, \Psi_0^{(2)}$.

For $n = 0$, we derive the following relation

$$\begin{aligned} \varepsilon^2\Psi_0 + \varepsilon^4\Psi_0^{(2)} = \Big((-2f_2\varepsilon^2 + A_1\varepsilon^4)|\Psi|^2 - 2f_2\varepsilon^4\left(\Psi_1^{(2)}(c.c.\Psi)\right.\\ \left. + \left(c.c.\Psi_1^{(2)}\right)\Psi\right) + A_2\varepsilon^4|\Psi|^4\Big) + h.o.t. \end{aligned} \tag{1.47a}$$

where $h.\,o.\,t$ = higher order terms and

$$A_1 = g_0 g_1, \quad A_2 = 10 f_3 f_2 - \frac{38}{9} f_2^3 + g_1 g_2 - \frac{11}{9} f_2 g_1^2 \tag{1.47b}$$

The obvious conclusion is

$$\Psi_0^{(2)} = (A_1|\Psi|^2 + A_2|\Psi|^4) - 2f_2(c.c.\Psi)\Psi_1^{(2)} + \Psi(c.c.\Psi)_1^{(2)} \tag{1.47c}$$

In a similar way, for $n = 2$, we obtain

$$\Psi_2^{(2)} = (B_1 + i\widetilde{B}_1)\Psi^2 + (B_2 + i\widetilde{B}_2)|\Psi|^2\Psi^2 + \frac{2}{3}(f_2 + ig_1)\Psi\Psi_1^{(2)} \tag{1.48}$$

with

$$B_1 = -\left(\frac{7}{18} g_0 g_1\right), \widetilde{B}_1 = \frac{2}{9} f_2 g_0 \tag{1.49a}$$

$$B_2 = -\frac{7}{18} f^{3_2} - \frac{5}{4} f_2 f_3 - \frac{19}{36} g_1 g_2 + \frac{7}{18} g^{2_1} f_2 \tag{1.49b}$$

$$\widetilde{B}_2 = -\frac{19}{288} g_1^3 - \frac{29}{12} f_2 g_2 + \frac{15}{4} f_3 g_1 - \frac{15}{4} f_2^2 g_1 \tag{1.49c}$$

If we neglect only terms of order ε^6 or higher, Eq. (1.33a–c) transforms into

$$\begin{aligned}
&+ \varepsilon^5 \left(-2i \frac{d\Psi_1^{(2)}}{dt} + (2f_2 + ig_1)\left(\Psi_0\Psi_1^{(2)} + \Psi_2\Psi_1^{(2)}\right) + (6f_3 + 2ig_2)|\Psi|^2\Psi_1^{(2)} \right.\\
&\quad \left. + (3f_3 + ig_2)\Psi^2\Psi_1^{(2)} + ig_0\Psi_1^{(2)} \right)\\
&+ \varepsilon^5 \left(\frac{d^2\Psi}{d\tau^2} + (2f_2 + ig_1)\left(\Psi_0^{(2)}\Psi + \Psi_2^{(2)}(c.c.\Psi) + (2f_2 + ig_1)\Psi_3(c.c.\Psi_2)\right)\right)\\
&+ \varepsilon^5 \left((6f_3 + 2ig_2)|\Psi_2|^2 + (3f_3 + ig_2)\Psi_0^2\Psi\right)\\
&+ \varepsilon^5 \left((6f_3 + 2ig_2)\Psi_2\Psi_0(c.c.\Psi) - g_0\frac{d\Psi}{d\tau} - \frac{g_2}{3}\frac{d|\Psi|^2\Psi}{d\tau} \right)\\
&+ \varepsilon^5 \left(\frac{g_1}{2}\left(\frac{(d(\Psi_0\Psi + \Psi_2(c.c.\Psi)))}{(d\tau)} \right)\right) + o(\varepsilon^7)
\end{aligned} \tag{1.50}$$

The term $\frac{d^2\Psi}{dt^2}$ in Eq. (1.50) can be eliminated taking into account that if we differentiate Eq. (1.36), we have

$$\frac{d^2\Psi}{dt^2} = \alpha_{1^2}\Psi + 4\alpha_1(\beta_1 + i\beta_2)|\Psi|^2\Psi + \left(3\beta_1^2 + 4i\beta_1\beta_2 - \beta_2^2\right)|\Psi|^4\Psi \tag{1.51}$$

Moreover, from (1.50), we see that it is necessary to consider Eq. (1.26) for $n = 3$

$$\Psi_3 = (C_1 + \mathrm{i}C_2)\Psi^3 \tag{1.52a}$$

where

$$C_1 = \frac{3f_3 + 2f_2^2 - 24g_1^2}{24}, C_2 = \frac{(3g_2 + 5f_2 g_1)}{24} \tag{1.52b}$$

If we use the abbreviation

$$\Psi_m = \Psi + e^2\Psi_1^{(2)} \tag{1.53a}$$

and Eq. (1.47c),(1.48), and (1.52a), then the final form of Eq. (1.46) is

$$\frac{d\Psi_m}{dt} = (\alpha_1 + i\widetilde{\alpha}_2)\Psi_m + (\beta_1 + \widetilde{\beta}_1 + i\beta_2 + i\widetilde{\beta}_2)|\Psi_m|^2\Psi_m + (\widetilde{g}_1 + i\widetilde{g}_2)|\Psi_m|^4\Psi_m \tag{1.53b}$$

with

$$\widetilde{\alpha}_2 = -\left(\frac{\alpha_1^2}{2}\right) + \frac{g_0\alpha_1}{2} \tag{1.54}$$

$$\widetilde{\beta}_1 = \frac{g_1}{2}A_1 + f_2\widetilde{B}_1 + \frac{g_1}{2}B - \frac{1}{24}g_0g_1^2 - \frac{3}{4}g_0f_3 + \frac{5}{6}g_0f_2^2 \tag{1.55}$$

$$\widetilde{\beta}_2 = \frac{g_1}{2}\widetilde{B}_1 - f_2B_1 - f_2A_1 - \frac{g_0g_2}{2} - \frac{3}{8}g_0g_1f_2 \tag{1.56}$$

$$\widetilde{\gamma}_1 = \frac{g_1}{2}A_2 + f_2\widetilde{B}_2 + \frac{g_1}{2}B_2 + \frac{11}{72}g_2g_1^2 + \frac{1}{36}f_2g_1^3 + \frac{63}{18}g_2f_2^2 - \frac{35}{36}g_1f_2^3 - \frac{5}{4}f_3g_2 + \frac{21}{24}g_1f_2f_3 \tag{1.57}$$

$$\widetilde{\gamma}_2 = \frac{g_1}{2}\widetilde{B}_2 - f_2B_2 - f_2A_2 - \frac{g_2^2}{8} - \frac{g_2g_1f_2}{8} + \frac{9}{8}f_2^3 + \frac{25}{18}f_2^4 - \frac{53}{6}f_3f_2^2 - \frac{11}{24}f_3g_1^2 + \frac{7}{18}g_1^2f_2^2 \tag{1.58}$$

and we arrive at the following modified system model

$$\frac{d\rho}{\mathrm{dt}} = \alpha_1\rho + (\beta_1 + \widetilde{\beta}_1)\rho^3 + \widetilde{\gamma}_1\rho^5 \tag{1.59}$$

$$\frac{d\theta}{dt} = (\beta_2 + \widetilde{\beta}_2)\rho^2 + \widetilde{\gamma}_2\rho^4 \tag{1.60}$$

The approximate solution up to the o(e^4)-th order is

$$\begin{aligned}
X(t) &= \varepsilon X_1(t) + \varepsilon^2X_2(t) + \varepsilon^3X_3(t) + \varepsilon^4X_4(t) + o(\varepsilon^5),\\
X_1(t) &= 2\rho(t)\cos(-t + \theta(t)),\\
X_2(t) &= -2f_2\rho^2(t) + \frac{2}{3}f_2\rho^2(t)\cos(-2t + 2\theta(t)) + \frac{2}{3}g_1\rho^2(t)\sin(-2t + 2\theta(t)),\\
X_3(t) &= 2C_1\rho^3(t)\cos(-3t + 3\theta(t)) + 2C_2\rho^3(t)\sin(-3t + 3\theta(t))\\
X_4(t) &= 2D_1\rho^4(t)\cos(-4t + 4\theta(t)) + 2D_2\rho^4(t)\sin(-4t + 4\theta(t)).
\end{aligned} \tag{1.61a}$$

where

$$D_1 = \frac{1}{15}\left(\frac{5}{18}f_2^3 + \frac{5}{4}f_2f_3 - \frac{31}{9}g_1^2f_2 - \frac{11}{6}g_1g_2\right) \tag{1.61b}$$

$$D_2 = \frac{1}{15}\left(\frac{43}{36}g_1f_2^3 + \frac{19}{12}f_2g_2 + \frac{3}{2}f_3g_1 - \frac{38}{9}g_1^3\right) \tag{1.61c}$$

We can calculate a more accurate expression for the limit cycle, and equations (1.44) and (1.45) become

$$\rho_E^M = \rho_E\left(1 + \frac{\widetilde{\gamma}_1\alpha_1 - \beta_1\widetilde{\beta}_1}{2\beta_1^2}\right) \tag{1.62}$$

$$\omega_E^M = \omega_E + \widetilde{\alpha}_2 + \widetilde{\beta}_2\left(\rho_E^M\right)^2 + \widetilde{\gamma}_2\rho_E^4 \tag{1.63}$$

If we examine the Eq. (1.59) carefully, we can easily understand that there is the possibility of two limit cycles and then of a qualitative change in the behavior of the oscillator with respect to the previsions of the first-order approximation. In fact, we suppose now that the dissipative coefficient g_0 is of order e^4 and g_2 is chosen in such a way that $g_2 - g_1 f_2$ is of order ε, then α_1 (see (1.37)) and β_1 (see (1.38)) are of order ε^4 and ε^2, respectively. For example, we can take $g_0 = 0.0001$, $g_1 = 1$, $g_2 = 1.01$, and $f_2 = 1$.

Taking into account that ρ must be of order e, then all terms in (39) have the same magnitude order and we can obtain for the equilibrium values of ρ two positive roots of order e and then two different limit cycles. Depending on the parameters, the larger limit cycle and the origin can be asymptotically stable and the smaller unstable or vice-versa.

By means of the variable change

$$t \rightarrow \frac{|\widetilde{g}_1|}{\beta^{2_1}} t,\ \rho \rightarrow \sqrt{\left|\frac{\beta_1}{\widetilde{g}_1}\right|}\rho \tag{1.64}$$

which implies

$$\alpha_1 \rightarrow \frac{|\widetilde{\gamma}_1|\alpha_1}{\beta^{2_1}} \tag{1.65}$$

we can always set $(\beta_1 + \widetilde{\beta}_1) = \pm 1, \widetilde{\gamma}_1 = \pm 1$.

There are four distinct cases:

(i) $(\beta_1 + \widetilde{\beta}_1) = 1, \widetilde{\gamma}_1 = 1$: for $\alpha_1 < 0$, there is an unstable limit cycle, and for $\alpha_1 > 0$, no limit cycle appears;

(ii) $(\beta_1 + \widetilde{\beta}_1) = -1, \widetilde{\gamma}_1 = -1$: for $\alpha_1 < 0$, there is no limit cycle, and for $\alpha_1 > 0$, only a stable limit cycle appears. Note that cases (i) and (ii) are connected by a simple temporal inversion, followed by the change of the sign of α_1.

(iii) $(\beta_1 + \widetilde{\beta}_1) = -1, \widetilde{\gamma}_1 = 1$: for $\alpha_1 < 0$, we have a larger stable limit cycle and a smaller unstable limit cycle; for $\alpha_1 > 0$, only the stable limit cycle is present (Figure 1.1).

(iv) $(\beta_1 + \widetilde{\beta}_1) = 1, \widetilde{\gamma}_1 = -1$: for $\alpha_1 < 0$, there is only an unstable limit cycle and for $\alpha_1 > 0$, a larger unstable limit cycle and a smaller stable limit cycle are present. Note that the last cases are also connected by a temporal inversion.

In all these cases, the origin is stable for $\alpha_1 < 0$ and unstable for $\alpha_1 > 0$.

1.4 Comparison with the Results of the Numerical Integration

To verify our analysis, we have computed a numerical solution of (1.26), by means of the standard Runge–Kutta method. A computer search was conducted to find these solutions, and their periodicity was verified. We have chosen the following set of parameters

$$f_2 = 2, f_3 = 1, g_0 = 0.01, g_1 = 2, g_2 = -2$$

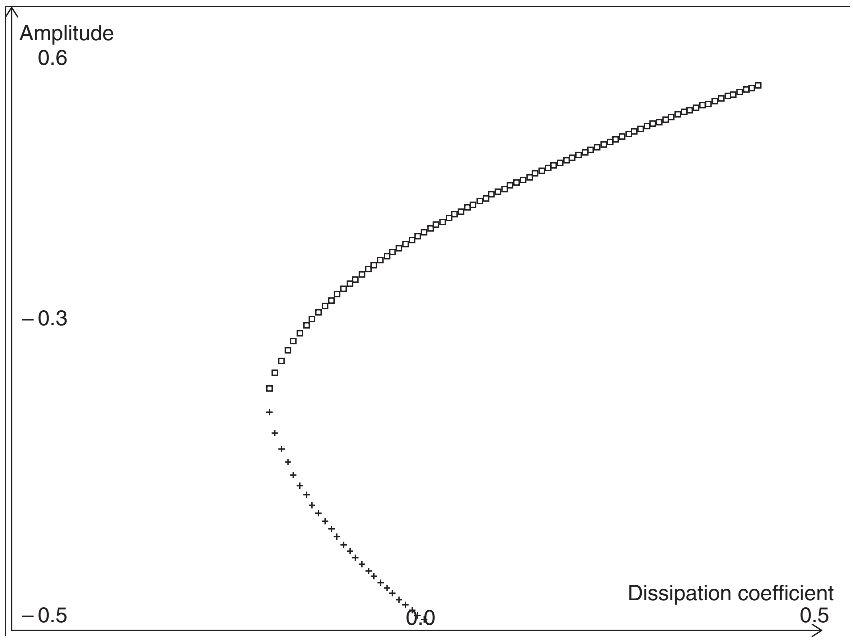

Figure 1.1 Dissipation (α_1)–response (ρ_E) space. Rectangles are stable limit cycles, and crosses represent unstable limit cycles.

We expect the appearance of a stable limit cycle with amplitude $\rho_E^M = 0.035$ (see equation (42) and case (iii) of the precedent section). In Figure 1.2, we show a comparison between the approximate solution (41) and the numerical solution: crosses represent the approximate solution and circles represent the numerical solution.

Only a cycle is represented, as the solution repeats itself one cycle after another. The agreement of the results appears to be excellent because the maximum difference is $6 \cdot 10^{-5}$ and the medium difference is $2 \cdot 10^{-5}$, i.e. of order e^5 as expected.

In Figure 1.3, we have increased the dissipation coefficient of an order of magnitude ($g_0 = 0.1$), and even in this case, we expect the appearance of a stable limit cycle with amplitude $\rho_E^M = 0.141$ (see Eq. (1.42) and case (iii) of the precedent section). The maximum difference between the approximate solution (1.61a–c) and the numerical solution is now $5 \cdot 10^{-2}$ and the medium difference is $2 \cdot 10^{-3}$.

The AP method is then a valid tool to approximate solutions of nonlinear oscillators with small dissipation coefficients.

1.5 External Excitation in Resonance with the Oscillator

In this section, we consider a nonlinear oscillator in resonance with an external force (see (1.27)). The AP method can be applied by following the same direction as in the preceding section. We obtain

$$\frac{d\Psi}{dt} = \alpha_1 \Psi + (\beta_1 + i\beta_2)|\Psi|^2\Psi + i\frac{f}{2} \tag{1.66}$$

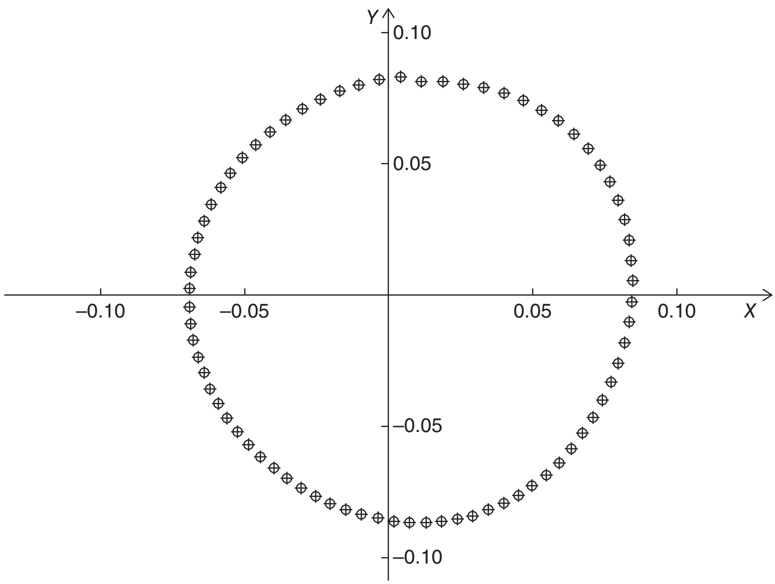

Figure 1.2 Phase space diagram ($X(t)$, $Y(t)$) with $Y(t) = \dot{X}(t)$ with $f_2 = 2, f_3 = 1, g_0 = 0.01$, $g_1 = 2$, and $g_2 = 2$. Circles are the numerical solution, and crosses represent the approximate solution.

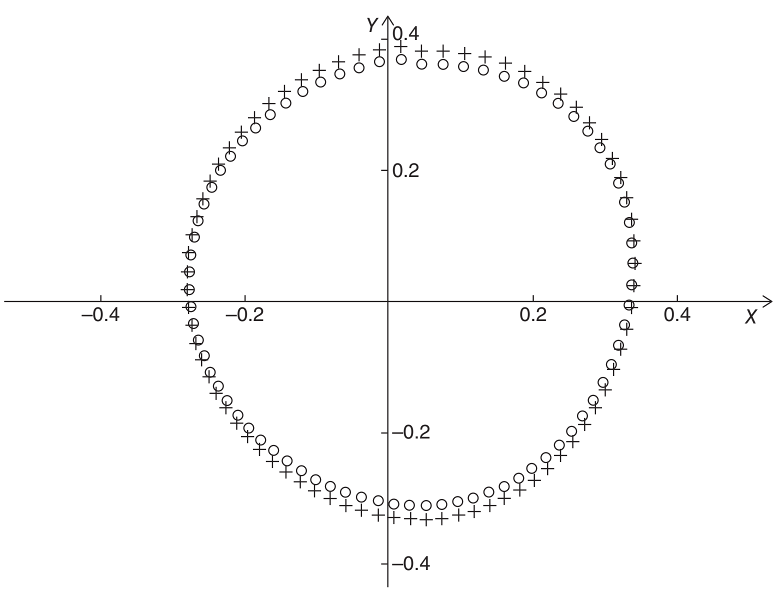

Figure 1.3 Phase space diagram ($X(t)$, $Y(t)$) with $Y(t) = \dot{X}(t)$ and $f_2 = 2, f_3 = 1, g_0 = 0.1$, $g_1 = 2$, and $g_2 = -2$. Circles are the numerical solution and crosses represent the approximate solution.

and then

$$\frac{d\rho}{dt} = \alpha_1\rho + \beta_1\rho^3 + \frac{f}{2}\sin\theta \tag{1.67}$$

$$\rho\frac{dJ}{dt} = \beta_2\rho^3 + \frac{f}{2}\cos\theta \tag{1.68}$$

We can easily determine the equilibrium points and their possible stability. By means of the variable change

$$t \to \frac{1}{|\alpha_1|}t, \rho \to \sqrt{\left|\frac{\alpha_1}{\beta_1}\right|}\rho \tag{1.69}$$

which implies

$$\beta_2 \to |\beta_1|\beta_2, \alpha_2 \to |\alpha_1|\alpha_2 \tag{1.70}$$

we can always set $\alpha_1 = \pm 1$, $\beta_1 = \pm 1$.

The equilibrium points ρ_0 must satisfy the equation

$$\mathrm{Br}^3 + 2\mathrm{sr}^2 + r - \frac{f^2}{4} = 0 \quad r = \rho_0^2 \tag{1.71a}$$

$$\theta_0 = \arctan\left(\frac{\alpha_1 + \beta_1 r}{\beta_2 r}\right) \tag{1.71b}$$

where

$$B = 1 + \beta_2^2 s = \alpha_1\beta_1 = \pm 1 \tag{1.71c}$$

Every equilibrium point of (1.71a–c) corresponds to a periodic solution of the starting system (1.27), and it is easily calculated by means of the standard formulas for the roots of a third-order equation.

The standard linearization method permits the computation of the Lyapunov exponents relative to each equilibrium point. They are

$$\lambda_1 = \alpha_1 + r(2\beta_1 + \sqrt{\Delta}), \lambda_2 = \alpha_1 + r(2\beta_1 - \sqrt{\Delta}) \tag{1.72a}$$

where

$$\Delta = 1 - 3\beta_2^2 \tag{1.72b}$$

We can now follow the stationary solutions of (1.71a–c) as the amplitude of the external force is varied, and at each step, along the curve that develops in external force–response space, the stability of the solution of (51) is determined by Eq. (52).

If $3B > 4$, we have only an equilibrium point (Figure 1.4). If $3B < 4$, $s > 0$, there is only an equilibrium point (Figure 1.5).If $3B < 4$, $s < 0$, we have one, two, or three equilibrium points (Figure 1.6); we observe a saddle–node fold (or cyclic fold) bifurcation, and the solution jumps up to a larger stable orbit as the amplitude of the external force is increased. A fold bifurcation corresponds to a vertical tangency in the external force–response space, where the derivative of the response with respect to the control parameter is infinite.

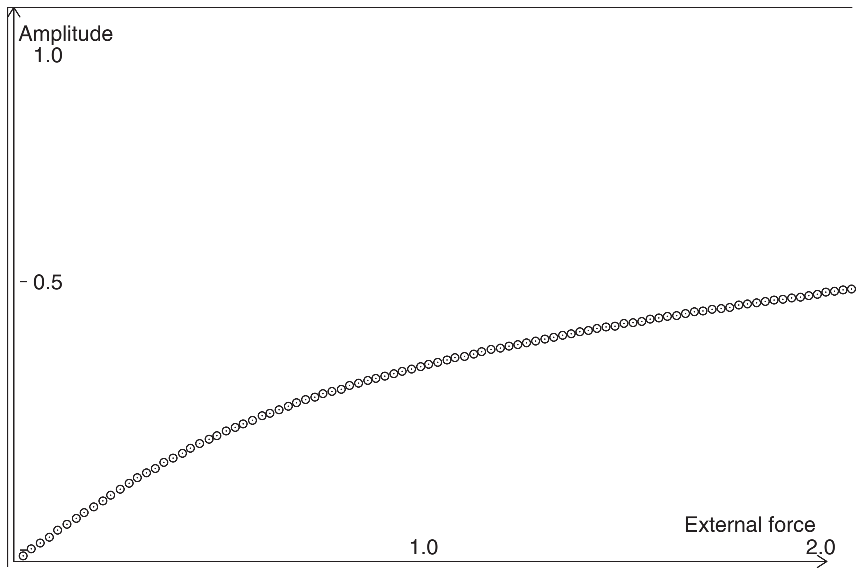

Figure 1.4 External force–response space. Circles are sources.

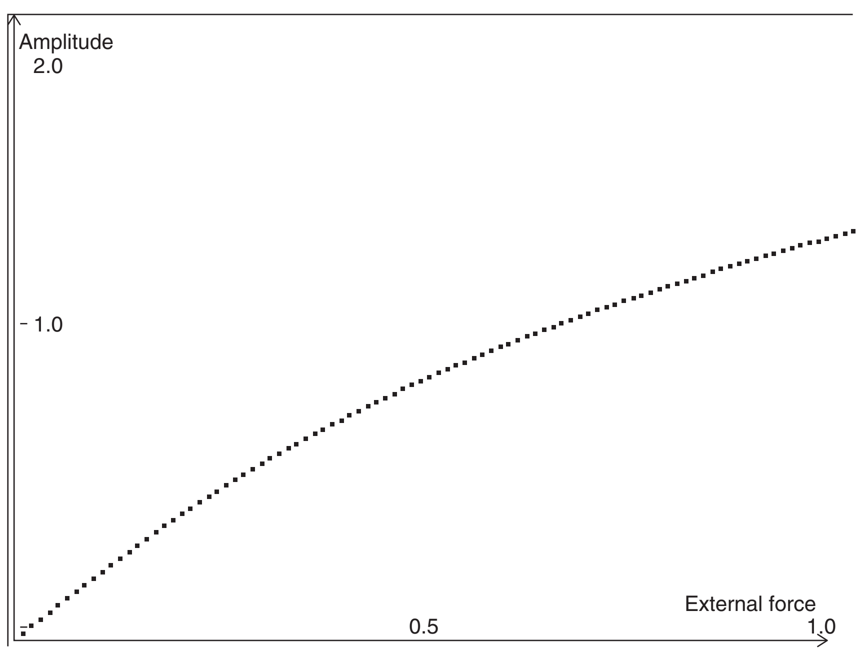

Figure 1.5 External force–response space. Rectangles are sinks.

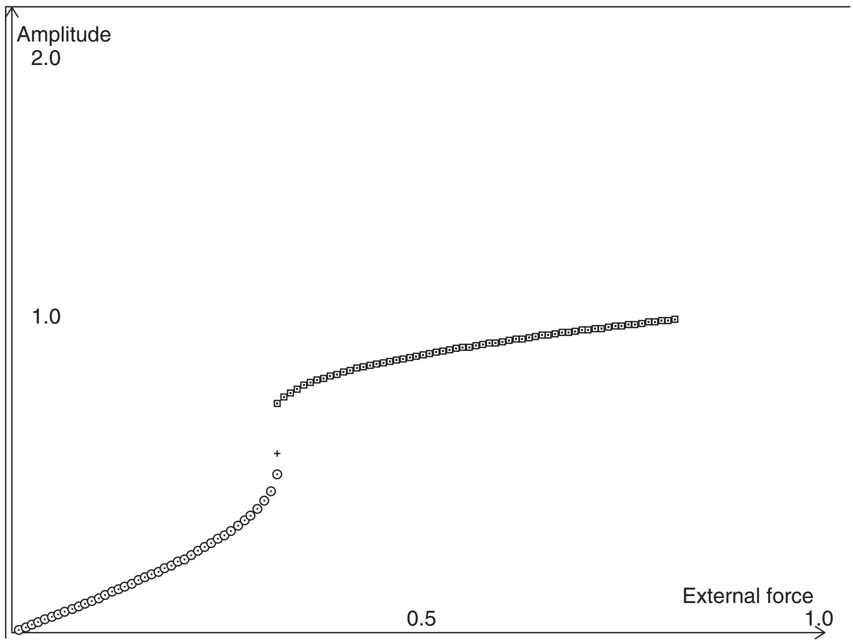

Figure 1.6 External force−response space. Rectangles are sinks, crosses represent saddle points, and circles stand for sources.

We now consider the system (1.67)–(1.68) when $\rho(t)$ is near (1.44). If the external excitation is sufficiently small, we obtain

$$\frac{d\rho}{dt} = -2\alpha_1(\rho - \rho_2) + \frac{f}{2}\sin(\Omega t + \theta_0) \tag{1.73}$$

with

$$\theta(t) = \Omega t + \theta_0 \tag{1.74a}$$

$$\Omega = \frac{\alpha_2\beta_1 - \beta_2\alpha_1}{\beta_1} \tag{1.74b}$$

where θ_0 depends on the initial conditions.

Equation (1.85) can be easily resolved and we get

$$\rho(t) = \rho_2 + (\rho_0 - \rho_2)\exp(-2\alpha_1 t) + \nu\frac{2\alpha_1\sin(\Omega t + \theta_0) - \Omega\cos(\Omega t + \theta_0) + (-2\alpha_1\sin(\theta_0) + \Omega\cos(\theta_0))\exp(-2\alpha_1 t)}{4\alpha^{2_1} + \Omega^2} \tag{1.75}$$

where ρ_0 depends on the initial conditions.

We conclude that the resulting motion is quasiperiodic, with the frequencies of 1 and Ω.

If we insert (1.74) and (1.75) into (1.43), we obtain the approximate solution up to the O(e^3)-th order.

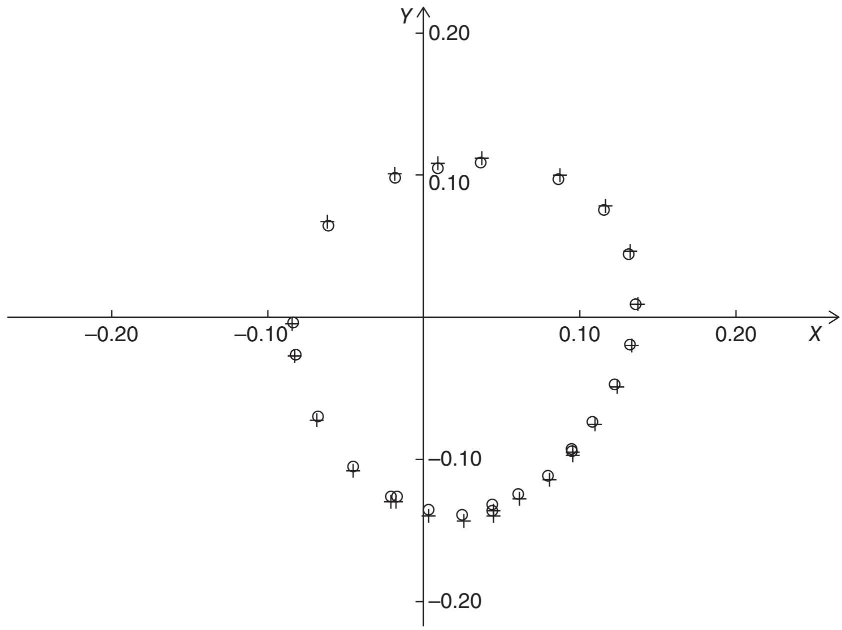

Figure 1.7 Associated map of the nonautonomous Eq. (1.27) with $f_2 = -1, f_3 = -1$, $g_0 = 0.02, g_1 = 1, g_2 = -3$, and $f = 3.5 \cdot 10^{-4}$. Crosses are the approximate solution, and circles represent the numerical solution.

In Figure 1.7, we show a comparison between the approximate solution (1.43) and the numerical solution. We represent the associated map of the nonautonomous Eq. (1.27), which is obtained with the values $(X(0),Y(0))$, $(X(T),Y(T))$, $(X(2T),Y(2T))$, …, where T is the period of the external excitation. Crosses represent the approximate solution and circles represent the numerical solution.

The closed curves reveal that the motion is quasiperiodic because of the presence of the frequency (1.74a). The agreement of the results is excellent because the maximum difference is 0.0075 and the medium difference is 0.0058, i.e. of order ε^3 as expected.

1.6 Conclusion

We have demonstrated the power of the AP method and how it produces useful approximate solutions. In particular, we have treated a class of strongly nonlinear oscillators subject to an external periodic force in resonance with the natural frequency of the oscillator.

We have found bifurcations and limit cycles, which are influenced by the presence of external excitation. In addition, we can observe a quasiperiodic motion, characterized by the combination of the natural frequency with a low frequency connected to the external excitation.

We indicate two possible extensions of the present chapter:

(i) study of subharmonic and superharmonic resonances for the system (6), with the observation of period-doubling bifurcations;
(ii) study of more complicated dynamical systems, such as three-dimensional systems or coupled oscillators, eventually subject to external excitations.