

# Table of Contents

## Chapter XIV. Evolution Problems: Cauchy Problems in $\mathbb{R}^n$

|                                                                                                                        |    |
|------------------------------------------------------------------------------------------------------------------------|----|
| Introduction . . . . .                                                                                                 | 1  |
| §1. The Ordinary Cauchy Problems in Finite Dimensional Spaces . . . .                                                  | 3  |
| 1. Linear Systems with Constant Coefficients . . . . .                                                                 | 4  |
| 2. Linear Systems with Non Constant Coefficients . . . . .                                                             | 6  |
| §2. Diffusion Equations. . . . .                                                                                       | 8  |
| 1. Setting of Problem . . . . .                                                                                        | 9  |
| 2. The Method of the Fourier Transform . . . . .                                                                       | 10 |
| 3. The Elementary Solution of the Heat Equation . . . . .                                                              | 15 |
| 4. Mathematical Properties of the Elementary Solution and the<br>Semigroup Associated with the Heat Operator . . . . . | 16 |
| §3. Wave Equations . . . . .                                                                                           | 21 |
| 1. Model Problem: The Wave Equation in $\mathbb{R}^n$ . . . . .                                                        | 21 |
| 2. The Euler–Poisson–Darboux Equation . . . . .                                                                        | 44 |
| 3. An Application of §2 and 3: Viscoelasticity . . . . .                                                               | 48 |
| §4. The Cauchy Problem for the Schrödinger Equation, Introduction . .                                                  | 53 |
| 1. Model Problem 1. The Case of Zero Potential . . . . .                                                               | 53 |
| 2. Model Problem 2. The Case of a Harmonic Oscillator. . . . .                                                         | 57 |
| §5. The Cauchy Problem for Evolution Equations Related to Convolution<br>Products . . . . .                            | 58 |
| 1. Setting of Problem . . . . .                                                                                        | 58 |
| 2. The Method of the Fourier Transform . . . . .                                                                       | 59 |
| 3. The Dirac Equation for a Free Particle . . . . .                                                                    | 63 |
| §6. An Abstract Cauchy Problem. Ovsyannikov's Theorem . . . . .                                                        | 66 |
| Review of Chapter XIV . . . . .                                                                                        | 72 |

## Chapter XV. Evolution Problems: The Method of Diagonalisation

|                                                                   |    |
|-------------------------------------------------------------------|----|
| Introduction . . . . .                                            | 73 |
| §1. The Fourier Method or the Method of Diagonalisation . . . . . | 74 |
| 1. The Case of the Space $\mathbb{R}^1$ ( $n = 1$ ). . . . .      | 74 |
| 2. The Case of Space Dimension $n = 2$ . . . . .                  | 94 |

|                                                                                                               |     |
|---------------------------------------------------------------------------------------------------------------|-----|
| 3. The Case of Arbitrary Dimension $n$ . . . . .                                                              | 99  |
| Review . . . . .                                                                                              | 103 |
| §2. Variations. The Method of Diagonalisation for an Operator Having Continuous Spectrum . . . . .            | 104 |
| 1. Review of Self-Adjoint Operators in Hilbert Spaces . . . . .                                               | 104 |
| 2. General Formulation of the Problem . . . . .                                                               | 104 |
| 3. A Simple Example of the Problem with Continuous Spectrum . . . . .                                         | 108 |
| §3. Examples of Application: The Diffusion Equation . . . . .                                                 | 112 |
| 1. Example of Application 1: The Monokinetic Diffusion Equation for Neutrons . . . . .                        | 112 |
| 2. Example of Application 2: The Age Equation in Problems of Slowing Down of Neutrons . . . . .               | 118 |
| 3. Example of Application 3: Heat Conduction . . . . .                                                        | 122 |
| §4. The Wave Equation: Mathematical Examples and Examples of Application . . . . .                            | 126 |
| 1. The Case of Dimension $n = 1$ . . . . .                                                                    | 126 |
| 2. The Case of Arbitrary Dimension $n$ . . . . .                                                              | 143 |
| 3. Examples of Applications for $n = 1$ . . . . .                                                             | 145 |
| 4. Examples of Applications for $n = 2$ . Vibrating Membranes. . . . .                                        | 156 |
| 5. Application to Elasticity; the Dynamics of Thin Homogeneous Beams . . . . .                                | 159 |
| §5. The Schrödinger Equation . . . . .                                                                        | 169 |
| 1. The Cauchy Problem for the Schrödinger Equation in a Domain $\Omega = ]0, 1[ \subset \mathbb{R}$ . . . . . | 170 |
| 2. A Harmonic Oscillator . . . . .                                                                            | 177 |
| Review . . . . .                                                                                              | 183 |
| §6. Application with an Operator Having a Continuous Spectrum: Example                                        | 184 |
| Review of Chapter XV . . . . .                                                                                | 186 |
| Appendix. Return to the Problem of Vibrating Strings . . . . .                                                | 186 |

## Chapter XVI. Evolution Problems: The Method of the Laplace Transform

|                                                                                           |     |
|-------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                    | 202 |
| §1. Laplace Transform of Distributions . . . . .                                          | 203 |
| 1. Study of the Set $I_f$ and Definition of the Laplace Transform. . . . .                | 204 |
| 2. Properties of the Laplace Transform . . . . .                                          | 210 |
| 3. Characterisation of Laplace Transforms of Distributions of $L_+(\mathbb{R})$ . . . . . | 212 |
| §2. Laplace Transform of Vector-valued Distributions . . . . .                            | 217 |
| 1. Distributions with Vector-valued Values . . . . .                                      | 218 |
| 2. Fourier and Laplace Transforms of Vector-valued Distributions . . . . .                | 222 |

|                                                                                                                         |     |
|-------------------------------------------------------------------------------------------------------------------------|-----|
| §3. Applications to First Order Evolution Problems . . . . .                                                            | 225 |
| 1. 'Vector-valued Distribution' Solutions of an Evolution Equation<br>of First Order in $t$ . . . . .                   | 225 |
| 2. The Method of Transposition. . . . .                                                                                 | 231 |
| 3. Application to First Order Evolution Equations. The Hilbert Space<br>Case. $L^2$ Solutions in Hilbert Space. . . . . | 233 |
| 4. The Case where $A$ is Defined by a Sesquilinear Form $a(u, v)$ . . . .                                               | 243 |
| §4. Evolution Problems of Second Order in $t$ . . . . .                                                                 | 251 |
| 1. Direct Method . . . . .                                                                                              | 251 |
| 2. Use of Symbolic Calculus . . . . .                                                                                   | 257 |
| Review . . . . .                                                                                                        | 261 |
| §5. Applications . . . . .                                                                                              | 261 |
| 1. Hydrodynamical Problems . . . . .                                                                                    | 261 |
| 2. A Problem of the Kinetics of Neutron Diffusion. . . . .                                                              | 265 |
| 3. Problems of Diffusion of an Electromagnetic Wave . . . . .                                                           | 267 |
| 4. Problems of Wave Propagation. . . . .                                                                                | 273 |
| 5. Viscoelastic Problems . . . . .                                                                                      | 280 |
| 6. A Problem Related to the Schrödinger Equation . . . . .                                                              | 290 |
| 7. A Problem Related to Causality, Analyticity and Dispersion Relations                                                 | 292 |
| 8. Remark 10. . . . .                                                                                                   | 295 |
| Review of Chapter XVI . . . . .                                                                                         | 296 |

## Chapter XVII. Evolution Problems: The Method of Semigroups

|                                                                                                         |     |
|---------------------------------------------------------------------------------------------------------|-----|
| Introduction . . . . .                                                                                  | 297 |
| <i>Part A. Study of Semigroups</i> . . . . .                                                            | 301 |
| §1. Definitions and Properties of Semigroups Acting in a Banach Space .                                 | 301 |
| 1. Definition of a Semigroup of Class $\mathcal{C}^0$ (Resp. of a Group). . . . .                       | 301 |
| 2. Basic Properties of Semigroups of Class $\mathcal{C}^0$ . . . . .                                    | 307 |
| §2. The Infinitesimal Generator of a Semigroup . . . . .                                                | 310 |
| 1. Examples . . . . .                                                                                   | 310 |
| 2. The Infinitesimal Generator of a Semigroup of Class $\mathcal{C}^0$ . . . . .                        | 315 |
| §3. The Hille–Yosida Theorem . . . . .                                                                  | 321 |
| 1. A Necessary Condition. . . . .                                                                       | 321 |
| 2. The Hille–Yosida Theorem. . . . .                                                                    | 323 |
| 3. Examples of Application of the Hille–Yosida Theorem . . . . .                                        | 327 |
| §4. The Case of Groups of Class $\mathcal{C}^0$ and Stone's Theorem. . . . .                            | 353 |
| 1. The Characterisation of the Infinitesimal Generator of a Group<br>of Class $\mathcal{C}^0$ . . . . . | 353 |
| 2. Unitary Groups of Class $\mathcal{C}^0$ . Stone's Theorem . . . . .                                  | 356 |

|                                                                                                                                             |     |
|---------------------------------------------------------------------------------------------------------------------------------------------|-----|
| 3. Applications of Stone's Theorem . . . . .                                                                                                | 357 |
| 4. Conservative Operators and Isometric Semigroups in Hilbert Space                                                                         | 362 |
| Review . . . . .                                                                                                                            | 365 |
| §5. Differentiable Semigroups . . . . .                                                                                                     | 365 |
| §6. Holomorphic Semigroups . . . . .                                                                                                        | 367 |
| §7. Compact Semigroups . . . . .                                                                                                            | 388 |
| 1. Definition and Principal Properties . . . . .                                                                                            | 388 |
| 2. Characterisation of Compact Semigroups . . . . .                                                                                         | 389 |
| 3. Examples of Compact Semigroups . . . . .                                                                                                 | 394 |
| <i>Part B. Cauchy Problems and Semigroups</i> . . . . .                                                                                     | 397 |
| §1. Cauchy Problems . . . . .                                                                                                               | 397 |
| §2. Asymptotic Behaviour of Solutions as $t \rightarrow +\infty$ . Conservation and<br>Dissipation in Evolution Equations . . . . .         | 406 |
| §3. Semigroups and Diffusion Problems . . . . .                                                                                             | 412 |
| §4. Groups and Evolution Equations . . . . .                                                                                                | 420 |
| 1. Wave Problems . . . . .                                                                                                                  | 420 |
| 2. Schrödinger Type Problems . . . . .                                                                                                      | 424 |
| 3. Weak Asymptotic Behaviour, for $t \rightarrow \pm\infty$ , of Solutions of<br>Wave Type of Schrödinger Type Problems . . . . .           | 426 |
| 4. The Cauchy Problem for Maxwell's Equations in an Open Set<br>$\Omega \subset \mathbb{R}^3$ . . . . .                                     | 433 |
| §5. Evolution Operators in Quantum Physics. The Liouville-von Neumann<br>Equation . . . . .                                                 | 439 |
| 1. Existence and Uniqueness of the Solution of the Cauchy Problem for<br>the Liouville-von Neumann Equation in the Space of Trace Operators | 439 |
| 2. The Evolution Equation of (Bounded) Observables in the Heisenberg<br>Representation . . . . .                                            | 446 |
| 3. Spectrum and Resolvent of the Operator $h$ . . . . .                                                                                     | 451 |
| §6. Trotter's Approximation Theorem . . . . .                                                                                               | 453 |
| 1. Convergence of Semigroups . . . . .                                                                                                      | 453 |
| 2. General Representation Theorem . . . . .                                                                                                 | 459 |
| Summary of Chapter XVII . . . . .                                                                                                           | 465 |

## Chapter XVIII. Evolution Problems: Variational Methods

|                                                    |     |
|----------------------------------------------------|-----|
| Introduction. Orientation . . . . .                | 467 |
| §1. Some Elements of Functional Analysis. . . . .  | 469 |
| 1. Review of Vector-valued Distributions . . . . . | 469 |
| 2. The Space $W(a, b; V, V')$ . . . . .            | 472 |

|                                                                                                                                   |     |
|-----------------------------------------------------------------------------------------------------------------------------------|-----|
| 3. The Spaces $W(a, b; X, Y)$ . . . . .                                                                                           | 479 |
| 4. Extension to Banach Space Framework . . . . .                                                                                  | 482 |
| 5. An Intermediate Derivatives Theorem . . . . .                                                                                  | 493 |
| 6. Bidual. Reflexivity. Weak Convergence and Weak * Convergence . . . . .                                                         | 499 |
| §2. Galerkin Approximation of a Hilbert Space . . . . .                                                                           | 503 |
| 1. Definition . . . . .                                                                                                           | 504 |
| 2. Examples . . . . .                                                                                                             | 504 |
| 3. The Outline of a Galerkin Method . . . . .                                                                                     | 507 |
| §3. Evolution Problems of First Order in $t$ . . . . .                                                                            | 509 |
| 1. Formulation of Problem $(P)$ . . . . .                                                                                         | 509 |
| 2. Uniqueness of the Solution of Problem $(P)$ . . . . .                                                                          | 512 |
| 3. Existence of a Solution of Problem $(P)$ . . . . .                                                                             | 513 |
| 4. Continuity with Respect to the Data . . . . .                                                                                  | 520 |
| 5. Appendix: Various Extensions – Liftings . . . . .                                                                              | 521 |
| §4. Problems of First Order in $t$ (Examples) . . . . .                                                                           | 523 |
| 1. Mathematical Example 1. Dirichlet Boundary Conditions . . . . .                                                                | 524 |
| 2. Mathematical Example 2. Neumann Boundary Conditions . . . . .                                                                  | 524 |
| 3. Mathematical Example 3. Mixed Dirichlet–Neumann Boundary<br>Conditions . . . . .                                               | 527 |
| 4. Mathematical Example 4. Bilinear Form Depending on Time $t$ . . . . .                                                          | 528 |
| 5. Evolution, Positivity and ‘Maximum’ of Solutions of Diffusion<br>Equations in $L^p(\Omega)$ , $1 \leq p \leq \infty$ . . . . . | 533 |
| 6. Mathematical Example 5. A Problem of Oblique Derivatives . . . . .                                                             | 539 |
| 7. Example of Application. The Neutron Diffusion Equation . . . . .                                                               | 542 |
| 8. A Stability Result . . . . .                                                                                                   | 548 |
| §5. Evolution Problems of Second Order in $t$ . . . . .                                                                           | 552 |
| 1. General Formulation of Problem $(P_1)$ . . . . .                                                                               | 552 |
| 2. Uniqueness in Problem $(P_1)$ . . . . .                                                                                        | 558 |
| 3. Existence of a Solution of Problem $(P_1)$ . . . . .                                                                           | 561 |
| 4. Continuity with Respect to the Data . . . . .                                                                                  | 566 |
| 5. Formulation of Problem $(P_2)$ . . . . .                                                                                       | 570 |
| §6. Problems of Second Order in $t$ . Examples . . . . .                                                                          | 581 |
| 1. Mathematical Example 1. . . . .                                                                                                | 581 |
| 2. Mathematical Example 2. . . . .                                                                                                | 582 |
| 3. Mathematical Example 3. . . . .                                                                                                | 583 |
| 4. Mathematical Example 4. . . . .                                                                                                | 587 |
| 5. Application Examples . . . . .                                                                                                 | 589 |
| §7. Other Types of Equation . . . . .                                                                                             | 620 |
| 1. Schrödinger Type Equations . . . . .                                                                                           | 620 |
| 2. Evolution Equations with Delay . . . . .                                                                                       | 643 |
| 3. Some Integro-Differential Equations . . . . .                                                                                  | 651 |
| 4. Optimal Control and Problems where the Unknowns are<br>Operators . . . . .                                                     | 662 |

5. The Problem of Coupled Parabolic-Hyperbolic Transmission . . . 670

6. The Method of 'Extension with Respect to a Parameter'. . . . . 676

Review of Chapter XVIII . . . . . 679

Bibliography . . . . . 680

Table of Notations . . . . . 686

Index. . . . . 702

Contents of Volumes 1-4, 6. . . . . 735