Contents

Chapter 1 Functions of Several Variables and Their Derivatives

1.1	Points and Points Sets in the	
	Plane and in Space	
	a. Sequences of points. Conver-	
	gence, 1 b. Sets of points in the	
1	plane, 3 c. The boundary of a set.	
	Closed and open sets, 6 d. Closure	
	as set of limit points, 9 e. Points	
	and sets of points in space, 9	
1.2	Functions of Several Independent	
	Variables	11
	a. Functions and their domains, 11	
	b. The simplest types of func-	
	tions, 12 c. Geometrical representa-	
	tion of functions, 13	
1.3	Continuity	17
	a. Definition, 17 b. The concept of	
	limit of a function of several vari-	
	ables, 19 c. The order to which a	
	function vanishes, 22	
1.4	The Partial Derivatives of a	
	Function	26
	a. Definition. Geometrical	
	representation, 26 b. Examples,	
	32 c. Continuity and the	
	existence of partial derivatives, 34	



	d. Change of the order of differentiation, 36	
1.5	The Differential of a Function and Its Geometrical Meaning a. The concept of differentiability, 40 b. Directional derivatives, 43 c. Geometric interpretation of differentiability, The tangent plane, 46 d. The total differential of a function, 49 e. Application to the calculus of errors, 52	40
1.6	Functions of Functions (Compound Functions) and the Introduction of New Independent Variables a. Compound functions. The chain rule, 53 b. Examples, 59 c. Change of independent variables, 60	53
1.7	The Mean Value Theorem and Taylor's Theorem for Functions of Several Variables a. Preliminary remarks about approximation by polynomials, 64 b. The mean value theorem, 66 c. Taylor's theorem for several independent variables, 68	64
1.8	Integrals of a Function Depending on a Parameter a. Examples and definitions, 71 b. Continuity and differentiability of an integral with respect to the parameter, 74 c. Interchange of integrations. Smoothing of functions, 80	71
1.9	Differentials and Line Integrals a. Linear differential forms, 82	82

ferential forms, 85 c. Dependence of line integrals on endpoints, 92 1.10 The Fundamental Theorem on Integrability of Linear Differential Forms a. Integration of total differentials, 95 b. Necessary conditions for line integrals to depend only on the end points, 96 c. Insufficiency of the integrability conditions, 98 d. Simply connected sets, 102 e. The fundamental theorem, 104	95
A.1. The Principle of the Point of Accumulation in Several Dimensions and Its Applications a. The principle of the point of accumulation, 107 b. Cauchy's convergence test. Compactness, 108 c. The Heine-Borel covering theorem, 109 d. An application of the Heine-Borel theorem to closed sets contains in open sets, 110.	107
A.2. Basic Properties of Continuous Functions	112
 A.3. Basic Notions of the Theory of Point Sets a. Sets and sub-sets, 113 b. Union and intersection of sets, 115 c. Applications to sets of points in the plane, 117. 	113
A.4. Homogeneous functions.	119

Chapter 2 Vectors, Matrices, Linear Transformations

2.1	Operations with Vectors a. Definition of vectors, 122 b. Geometric representation of vectors, 124 c. Length of vectors. Angles between directions, 127 d. Scalar products of vectors, 131 e. Equation of hyperplanes in vector form, 133 f. Linear dependence of vectors and systems of linear equations, 136	122
2.2	Matrices and Linear Transforma- tions a. Change of base. Linear spaces, 143 b. Matrices, 146 c. Opera- tions with matrices, 150 d. Square matrices. The reciprocal of a mat- rix. Orthogonal matrices. 153	143
2.3	Determinants a. Determinants of second and third order, 159 b. Linear and multilinear forms of vectors, 163 c. Alternating multilinear forms. Definition of determinants, 166 d. Principal properties of determinants, 171 e. Application of determinants to systems of linear equations. 175	159
2.4	Geometrical Interpretation of Determinants a. Vector products and volumes of parallelepipeds in three-dimensional space, 180 b. Expansion of a determinant with respect to a column.	180

Vector products in higher dimensions, 187 c. Areas of parallelograms and volumes of parallelepipeds in

. , .	higher dimensions, 190 d. Orientation of parallelepipeds in <i>n</i> -dimensional space, 195 e. Orientation of planes and hyperplanes, 200 f. Change of volume of parallelepipeds in linear transformations, 201	
2.5	Vector Notions in Analysis a. Vector fields, 204 b. Gradient of a scalar, 205 c. Divergence and curl of a vector field, 208 d. Families of vectors. Application to the theory of curves in space and to motion of particles, 211	204
•	evelopments and Applications the Differential Calculus	
3.1	Implicit Functions a. General remarks, 218 b. Geometrical interpretation, 219 c. The implicit function theorem, 221 d. Proof of the implicit function theorem, 225 e. The implicit function theorem for more than two independent variables, 228	218
3.2	Curves and Surfaces in Implicit Form a. Plane curves in implicit form, 230 b. Singular points of curves, 236 c. Implicit representation of surfaces, 238	230
3.3	Systems of Functions, Transformations, and Mappings a. General remarks, 241 b. Curvilinear coordinates, 246 c. Extension to more than two independent variables, 249 d. Differentiation	241

: 4

	252 e. Symbolic product of mappings, 257 f. General theorem on the inversion of transformations and of systems of implicit functions. Decomposition into primitive mappings, 261 g. Alternate construction of the inverse mapping by the method of successive approximations, 266 h. Dependent functions, 268 i. Concluding remarks, 275	
3.4	Applications a. Elements of the theory of surfaces, 278 b. Conformal transformation in general, 289	278
3.5	Families of Curves, Families of Surfaces, and Their Envelopes a. General remarks, 290 b. Envelopes of one-parameter families of curves, 292 c. Examples, 296 d. Endevelopes of families of surfaces, 303	290
3.6	Alternating Differential Forms a. Definition of alternating dif- ferential forms, 307 b. Sums and products of differential forms, 310 c. Exterior derivatives of differ- ential forms, 312 d. Exterior differential forms in arbitrary coordinates, 316	307
3.7	Maxima and Minima a. Necessary conditions, 325 b. Examples, 327 c. Maxima and minima with subsidiary conditions, 330 d. Proof of the method of undetermined multipliers in the simplest case, 334 e. Generalization of the method of undetermined multipliers, 337 f. Examples 340	325

		Contents	XV
	APF	PENDIX	
	A.1	Sufficient Conditions for Extreme Values	345
	A.2	Numbers of Critical Points Re- lated to Indices of a Vector Field	352
	A.3	Singular Points of Plane Curves	360
	A.4	Singular Points of Surfaces	362
	A. 5	Connection Between Euler's and Lagrange's Representation of the motion of a Fluid	363
	A.6	Tangential Representation of a Closed Curve and the Isoperi- metric Inequality	365
Chapter 4	Mu	ultiple Integrals	
	4.1	Areas in the Plane a. Definition of the Jordan measure of area, 367 b. A set that does not have an area, 370 c. Rules for operations with areas, 372	367
	4.2	Double Integrals a. The double integral as a volume, 374 b. The general analytic concept of the integral, 376 c. Examples, 379 d. Notation. Extensions. Fundamental rules, 381 e. Integral estimates and the mean value theorem, 383	374

4.3 Integrals over Regions in three and more Dimensions

385

4.4	Space Differentiation. Mass and Density	386
4.5	Reduction of the Multiple Integral to Repeated Single Integrals a. Integrals over a rectangle, 388 b. Change of order of integration. Differentiation under the integral sign, 390 c. Reduction of double integrals to single integrals for more general regions, 392 d. Ex- tension of the results to regions in several dimensions, 397	388
4.6	Transformation of Multiple Integrals a. Transformation of integrals in the plane, 398 b. Regions of more than two dimensions, 403	398
4.7	Improper Multiple Integrals a. Improper integrals of functions over bounded sets, 407 b. Proof of the general convergence theorem for improper integrals, 411 c. Integrals over unbounded regions, 414	406
4.8	Geometrical Applications a. Elementary calculation of volumes, 417 b. General remarks on the calculation of volumes. Solids of revolution. Volumes in spherical coordinates, 419 c. Area of a curved surface, 421	417
4.9	Physical Applications a. Moments and center of mass, 431 b. Moments of inertia, 433 c. The compound pendulum, 436 d. Potential of attracting masses, 438	431

4.10	Coordinates	445
	a. Resolution of multiple integrals, 445 b. Application to areas swept out by moving curves and volumes swept out by moving surfaces. Guldin's formula. The polar planimeter, 448	
4.11	Volumes and Surface Areas in Any Number of Dimensions a. Surface areas and surface integrals in more than three dimensions, 453 b. Area and volume of the <i>n</i> -dimensional sphere, 455 c. Generalizations. Parametric Representations, 459	458
4.12	Improper Single Integrals as Functions of a Parameter a. Uniform convergence. Continuous dependence on the parameter, 462 b. Integration and differentiation of improper integrals with respect to a parameter, 466 c. Examples, 469 d. Evaluation of Fresnel's integrals, 473	462
4.13	The Fourier Integral a. Introduction, 476 b. Examples, 479 c. Proof of Fourier's integral theorem, 481 d. Rate of conver- gence in Fourier's integral theorem, 485 e. Parseval's identity for Fourier transforms, 488 f. The Fourier transformation for func- tions of several variables, 490	476
4.14	The Eulerian Integrals (Gamma Function) a. Definition and functional equa-	497

	tion, 497 b. Convex functions. Proof of Bohr and Mollerup's theorem, 499 c. The infinite products for the gamma function, 503 d. The nextensio theorem, 507 e. The beta function, 508 f. Differentiation and integration of fractional order. Abel's integral equation, 511	
	PENDIX: DETAILED ANALYSIS OF E PROCESS OF INTEGRATION	
A.1	Area a. Subdivisions of the plane and the corresponding inner and outer areas, 515 b. Jordan-measurable sets and their areas, 517 c. Basic properties of areas, 519	515
A.2	Integrals of Functions of Several Variables a. Definition of the integral of a function $f(x, y)$, 524 b. Integrability of continuous functions and integrals over sets, 526 c. Basic rules for multiple integrals, 528 d. Reduction of multiple integrals to repeated single integrals, 531	524
A.3	Transformation of Areas and Integrals a. Mappings of sets, 534 b. Trans formation of multiple integrals, 539	534
A.4	Note on the Definition of the Area of a Curved Surface	540

Chapter 5 Relations Between Surface and Volume Integrals

5.1	Connection Between Line Integrals and Double Integrals in the Plane (The Integral Theorems of Gauss, Stokes, and Green)	548
5.2	Vector Form of the Divergence Theorem. Stokes's Theorem	551
5.3	Formula for Integration by Parts in Two Dimensions. Green's Theorem	556
5.4	The Divergence Theorem Applied to the Transformation of Double Integrals a. The case of 1-1 mappings, 558 b. Transformation of integrals and degree of mapping, 561	558
5.5	Area Differentiation. Transformation of Δu to Polar Coordinates	565
5.6	Interpretation of the Formulae of Gauss and Stokes by Two- Dimensional Flows	569
5.7	Orientation of Surfaces a. Orientation of two-dimensional surfaces in three-space, 575 b. Orientation of curves on oriented surfaces, 587	575
5. 8	Integrals of Differential Forms and of Scalars over Surfaces a. Double integrals over oriented plane regions, 589 b. Surface	589

integrals of second-order differential
forms, 592 c. Relation between
integrals of differential forms over
oriented surfaces to integrals of
scalars over unoriented surfaces,
594

5.9	Gauss's and Green's Theorems in Space a. Gauss's theorem, 597 b. Application of Gauss's theorem to fluid flow, 602 c. Gauss's theorem applied to space forces and surface forces, 605 d. Integration by parts and Green's theorem in three dimensions, 607 e. Application of Green's theorem to the transformation of ΔU to spherical coordinates, 608	597
5.10	Stokes's Theorem in Space a. Statement and proof of the theorem, 611 b. Interpretation of Stokes's theorem, 615	611
5.11	Integral Identities in Higher Dimensions	622
SUF	PENDIX: GENERAL THEORY OF RFACES AND OF SURFACE EGALS	
A.1	Surfaces and Surface Integrals in Three dimensions a. Elementary surfaces, 624 b. Integral of a function over an elementary surface, 627 c.Oriented elementary surfaces, 629 d. Simple surfaces, 631 e. Partitions of unity	624

and integrals over simple surfaces,

634

		Contents	XX
··	A.2	The Divergence Theorem a. Statement of the theorem and its invariance, 637 b. Proof of the theorem, 639	63'
	A.3	Stokes's Theorem	642
•	A.4	Surfaces and Surface Integrals in Euclidean Spaces of Higher Dimensions a. Elementary surfaces, 645 b. Integral of a differential form over an oriented elementary surface, 647 c. Simple m-dimensional surfaces, 648	64
	A.5	Integrals over Simple Surfaces, Gauss's Divergence Theorem, and the General Stokes Formula in Higher Dimensions	651
Chapter 6	Dif	ferential Equations	
	6.1	The Differential Equations for the Motion of a Particle in Three Dimensions a. The equations of motion, 654 b. The principle of conservation of energy, 656 c. Equilibrium. Stability, 659 d. Small oscillations about a position of equilibrium, 661 e. Planetary motion, 665 f. Boundary value problems. The loaded cable and the loaded beam, 672	654
	6.2	The General Linear Differential Equation of the First Order a. Separation of variables, 678 b. The linear first-order equation, 680	678

Linear Differential Equations of Higher Order a. Principle of superposition. General solutions, 683 b. Homogeneous differential equations of the second second order, 688 c. The non- homogeneous differential equations. Method of variation of parameters, 691	683
General Differential Equations of the First Order a. Geometrical interpretation, 697 b. The differential equation of a family of curves. Singular solutions. Orthogonal trajectories, 699 c. Theorem of the existence and uniqueness of the solution, 702	697
Systems of Differential Equations and Differential Equations of Higher Order	709
Integration by the Method of Undermined Coefficients	711
The Potential of Attracting Charges and Laplace's Equation a. Potentials of mass distributions, 713 b. The differential equation	713
of the potential, 718 c. Uniform double layers, 719 d. The mean value theorem, 722 e. Boundary value problem for the circle. Poisson's integral, 724	
	a. Principle of superposition. General solutions, 683 b. Homogeneous differential equations of the second second order, 688 c. The nonhomogeneous differential equations. Method of variation of parameters, 691 General Differential Equations of the First Order a. Geometrical interpretation, 697 b. The differential equation of a family of curves. Singular solutions. Orthogonal trajectories, 699 c. Theorem of the existence and uniqueness of the solution, 702 Systems of Differential Equations and Differential Equations of Higher Order Integration by the Method of Undermined Coefficients The Potential of Attracting Charges and Laplace's Equation a. Potentials of mass distributions,

	Y	c. Maxwell's equations in free space, 731	
Chapter 7	Calculus of Variations		
	7.1	Functions and Their Extrema	737
	7.2	Necessary conditions for Extreme Values of a Functional a. Vanishing of the first variation, 741 b. Deduction of Euler's differential equation, 743 c. Proofs of the fundamental lemmas, 747 d. Solution of Euler's differential equation in special cases. Examples, 748 e. Identical vanishing of Euler's expression, 752	741
	7.3	Generalizations a. Integrals with more than one argument function, 753 b. Examples, 755 c. Hamilton's principle. Lagrange's equations, 757 d. Integrals involving higher derivatives, 759 e. Several independent variables, 760	753
	7.4	Problems Involving Subsidiary Conditions. Lagrange Multi- pliers a. Ordinary subsidiary conditions, 762 b. Other types of subsidiary conditions, 765	762
Chapter 8	apter 8 Functions of a Complex Variable		le
	8.1	Complex Functions Represented by Power Series a. Limits and infinite series with complex terms, 769 b. Power	769

in three-dimensional space, 728

1

· , ; '

•		series, 772 c. Differentiation and integration of power series, 773 d. Examples of power series, 776	
	8.2	Foundations of the General Theory of Functions of a Complex Variable a. The postulate of differentiability, 778 b. The simplest operations of the differential calculus, 782 c. Conformal transformation. Inverse functions, 785	778
5	8.3	The Integration of Analytic Functions a. Definition of the integral, 787 b. Cauchy's theorem, 789 c. Applications. The logarithm, the exponential function, and the general power function, 792	787
•	8.4	Cauchy's Formula and Its Applications a. Cauchy's formula, 797 b. Expansion of analytic functions in power series, 799 c. The theory of functions and potential theory, 802 d. The converse of Cauchy's theorem, 803 e. Zeros, poles, and residues of an analytic function, 803	797
	8.5	Applications to Complex Integration (Contour Integration) a. Proof of the formula (8.22), 807 b. Proof of the formula (8.22), 808 c. Application of the theorem of residues to the integration of rational functions, 809 d. The theorem of residues and linear differential equations with constant coefficients, 812	807

		Contents	XXV
8.6	Many-Valued Functions and Analytic Extension		814
List of Biograp	hical Dates		941
Index			943