Chapter 8

Integral Calculus

8

8

8 Integral Calculus

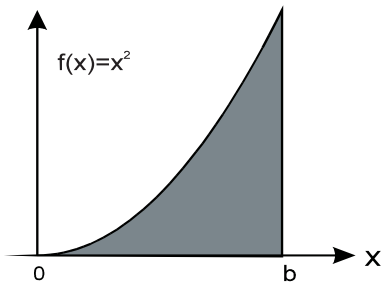
The integral operation, like the derivative of a function, is motivated e.g. by the physical description of a motion (velocity and acceleration). Many advanced topics in this volume such as the integral transform like Laplace transform and Fourier transform but also solution formulas for the first order differential equations are based on the integral notion. Integration is also important for the calculation of areas, volume of bodies, center of gravity calculations etc.

8.1 Integration

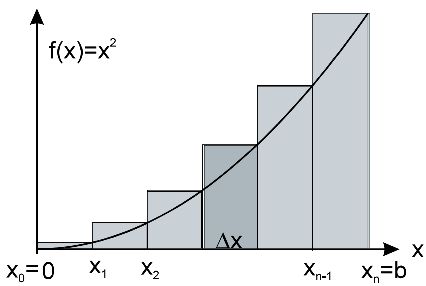
We start with a geometrical problem: Given is a non-negative function $f(x)$. How large is the area that encloses the curve with the x-axis within the interval $[a, b]$?

Example 8.1. To determine the area A_0^b beyond the graph of the function $f(x) = x^2$ in $[0, b]$ we divide the interval $[0, b]$ into n sub-intervals by an even division Z_n

$$x_0 = 0,\ x_1 = \frac{b}{n},\ x_2 = 2\frac{b}{n}, \dots,\ x_{n-1} = (n-1)\,\frac{b}{n},\ x_n = b.$$



Area under curve



Approximation by rectangles

For each sub-interval with interval length $\Delta x = x_k - x_{k-1} = \frac{b}{n}$ we select the right corner point $x_k = k \cdot \Delta x$ and evaluate the function

$$f(x_k) = x_k^2 = (k \cdot \Delta x)^2 .$$

The area of the corresponding rectangle is

$$\Delta x \cdot f(x_k) = \Delta x \cdot k^2 \cdot \Delta x^2.$$

Then, we sum up all rectangle surfaces to

$$S_n = \Delta x\, f(x_1) + \Delta x\, f(x_2) + \ldots + \Delta x\, f(x_n)$$
$$= \sum_{k=1}^{n} \Delta x\, f(x_k) = (\Delta x)^3 \sum_{k=1}^{n} k^2.$$

According to Volume 1 (in Section: Mathematical Induction) this sum can be simplified by

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}\, n\,(n+1)\,(2n+1)$$

$$\Rightarrow S_n = (\Delta x)^3 \frac{1}{6}\, n\,(n+1)\,(2n+1) = \frac{b^3}{n^3}\frac{1}{6}\, n\,(n+1)\,(2n+1) \xrightarrow{n\to\infty} \frac{b^3}{3}.$$

By refining the decomposition Z_n of the interval, the area beneath $f(x) = x^2$ is approximated with any precision. For $n \to \infty$ the *subtotal* S_n converges to the area below the graph of x^2: $A_0^b = \frac{b^3}{3}$. □

Step function

Stufenfunktion

The procedure from this tutorial is generalized by applying the following construction to calculate the area below any $f(x)$ curve with the x axis: First, the curve is approximated by a piecewise constant function (step function). All rectangular surfaces are summed up to give an approximation for the surface below the curve. Finally, by increasing the number of subdivisions the function is approximated more and more precisely by the step function. Finally, the area under the curve is approximated by the sum of the rectangular areas. The following definition for the definite integral is more precise:

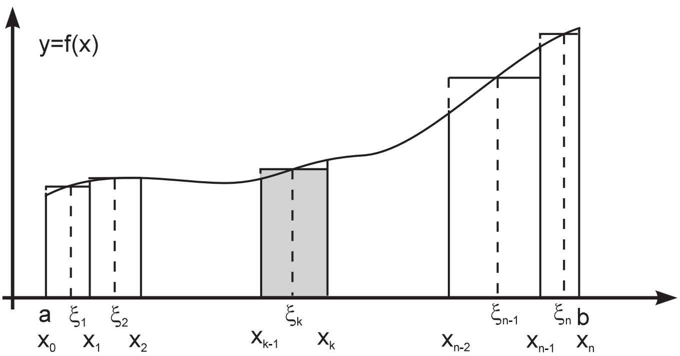


Figure 8.1. Subtotal

Definite Integral

Bestimmtes Integral

Definition: (Definite integral; Riemann Integral)
Given is a continuous function $f : [a, b] \to \mathbb{R}$ *with* $y = f(x)$.

(1) *Let* Z_n *be an equidistant subdivision of the interval* $a \le x \le b$ *in* n *sub-intervals*

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

The lengths are $\Delta x_k = x_k - x_{k-1} = \frac{b-a}{n}$. *Let* $\xi_k \in [x_{k-1}, x_k]$ *be any intermediate point in the interval. Then*

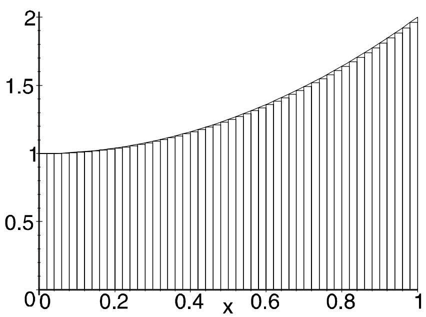
$$S_n = \sum_{k=1}^{n} \Delta x_k \, f(\xi_k)$$

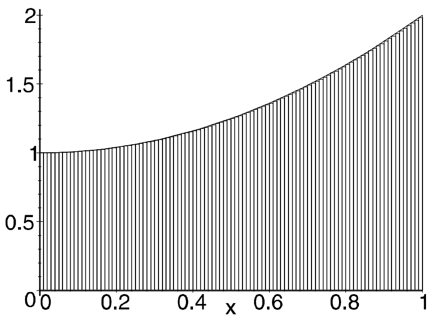
is called the Riemann subtotal with respect to the decomposition Z_n.

(2) *Under the definite integral (Riemann-Integral) of the continuous function* f *in the range from* $x = a$ *to* $x = b$ *the limit of the Riemann subtotal* S_n *for* $n \to \infty$ *is defined:*

$$\int_a^b f(x)\, dx := \lim_{n \to \infty} \sum_{k=1}^{n} \Delta x_k \, f(\xi_k).$$

Visualization: More illustrative than any precise mathematical definition is the descriptive interpretation. The animation shows the transition from the subtotal to the integral by increasing the number of subdivisions of the $[a, b]$ interval. This animation suggests the convergence from the discrete subtotal to the integral. Shown in the two following figures are the values of the integral $\int_0^1 (x^2 + 1)\, dx$ for a subdivisions $N = 50$ (left) and $N = 100$ (right).





Remarks:

(1) This integral concept is introduced according to the mathematician *Riemann* (1826 - 1866) and therefore called Riemann's Integral. Since we deal exclusively with this integral, we refer briefly to it as *the* integral.

(2) If f is continuous, then for any subdivision Z_n and any choice of $\xi_k \in [x_{k-1}, x_k]$, the subtotal converges with $\Delta x_k = x_k - x_{k-1} \overset{n\to\infty}{\longrightarrow} 0$ towards the same value. We say the integral is *well defined*.

(3) More generally, a function is called *integratable* if for any given subdivision Z_n and any given choice of $\xi_k \in [x_{k-1}, x_k]$ the subtotal converges towards the same value.

(4) This algebraic definition of the integral corresponds exactly to the procedure for the area calculation of the initial example for a non-negative function f. However, the algebraic definition is more general and thus goes beyond the area calculation.

(5) Common names for the symbols occurring in the given integral $\int_a^b f(x)\, dx$ are:

x: *Integration variable;* $f(x)$: *Integrand;*
a: *Lower bound;* b: *Upper bound.*

Example 8.2. A mass moves with the speed $v(t)$ along the x-axis. At time $t = 0$ it is located at position $x = 0$. Find the distance $x(T)$ at time $t = T$.

If the velocity is constant, $v(t) = v_0$, then the distance is $x(T) = v_0\, T$. For a non-constant velocity $v(t)$ we divide the time interval $[0, T]$ into sub-intervals

$$0 = t_0 < t_1 < t_2 < \ldots < t_n = T,$$

such that $v(t)$ behaves approximately constant in each sub-interval:

$$v(t) \approx v(t_{k-1}) \qquad \text{for} \quad t \in [t_{k-1}, t_k] \quad k = 1, \ldots, n.$$

Then, the distance $x(t_k)$ is calculated at time $t = t_k \quad (k = 1,\ 2,\ \ldots,\ n)$ approximately by

$$\begin{aligned}
x(t_1) &\approx (t_1 - t_0) \cdot v(t_0) = \Delta t_1\, v(t_0) \\
x(t_2) &\approx x(t_1) + (t_2 - t_1) \cdot v(t_1) = x(t_1) + \Delta t_2\, v(t_1) \\
x(t_3) &\approx x(t_2) + (t_3 - t_2) \cdot v(t_2) = \Delta t_1\, v(t_0) + \Delta t_2\, v(t_1) + \Delta t_3\, v(t_2) \\
&\vdots \\
x(t_n) &= x(T) \approx \sum_{k=1}^{n} \Delta t_k\, v(t_{k-1})\,.
\end{aligned}$$

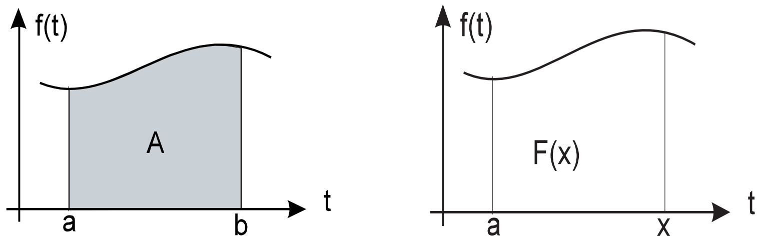
The approximate value obtained for $x(T)$ is thus the Riemann subtotal S_n. The exact value of the distance is

$$x(T) = \int_0^T v(t)\, dt.$$

□

The Indefinite Integral

For positive function the definite integral $\int_a^b f(t)\ dt$ represents the area between the curve $f(t)$ and the time axis. If we consider the lower bound to be fixed, the upper bound to be flexible, the integral value depends on this upper bound.



Definite integral   Integral function

To symbolize the dependency on the upper limit, we replace b with x and get a function $F(x) = \int_a^x f(t)\ dt$:

Definition: (Indefinite Integral, Integral Function)
The indefinite integral

$$F(x) := \int_a^x f(t)\ dt$$

is the integral function $F(x)$, for which the upper bound of the integral is variable.

Indefinite Integral
Uneigentliches Integral

Integral Function
Integralfunktion

For the indefinite integral $F(x) = \int_a^x f(t)\ dt$, the area between the function $f(t)$ and the t-axis depends on the upper bound.

Example 8.3. If we select $f(t) = t^2$ and $a = 0$ then the corresponding integral function is according to Example 8.1 the function $F : \mathbb{R} \to \mathbb{R}$ with $F(x) = \frac{x^3}{3}$. Note that there is a relation between the integral function and its integrand: $F'(x) = f(x)$. This relationship is very general, as shown in Section 8.2. □

Numerical Integration

Unfortunately we have to state, that even relatively simple functions can not be integrated elementarily. Examples are e^{-x^2} or $\frac{\sin x}{x}$. In these cases it is necessary to use numerical methods: According to the definition of the definite integral, we divide the interval $[a, b]$ into n sub-intervals $[x_i, x_{i+1}]$ with the interval length $h := \frac{b-a}{n}$ and set

$$x_0 = a\,; \quad x_{i+1} = x_i + h \quad (i = 0, \ldots, n-1)\,; \quad x_n = b.$$

If the function $f(x)$ to be integrated is replaced in each interval $[x_i, x_{i+1}]$ by a constant $f(\xi_i)$, $\xi_i \in [x_i, x_{i+1}]$, the integral is approximated by the subtotal

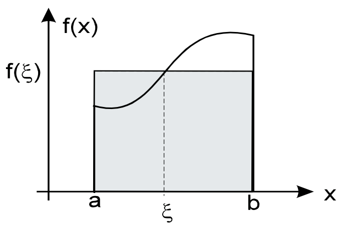
$$S_n \approx \sum_{i=0}^{n-1} A_i = \sum_{i=0}^{n-1} f(\xi_i)\,(x_{i+1} - x_i) = h \sum_{i=0}^{n-1} f(\xi_i)\,.$$

Thus, the simplest approximation is

$$\int_a^b f(x)\,dx \approx h\,(f(\xi_0) + f(\xi_1) + \ldots + f(\xi_{n-1}))\,.$$

8.2

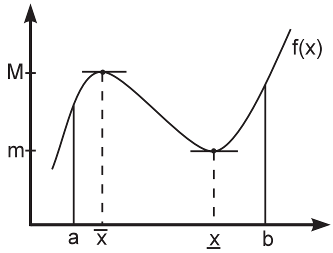
8.2 Fundamental Theorem of Calculus



As complicated as the construction of the definite integral may look, it turns out that in many cases the calculation becomes very simple. This fact is due to the relationship between the derivative of the integral function and the integrand, which is now investigated. We first present a generalization of the mean value theorem of the differential calculus, which states that the area below a curve $f(x)$ can be replaced by an area-equivalent rectangular area with the same base side and adapted height $f(\xi)$. Where $f(\xi)$ means *integral mean value* of the function f in the interval $[a, b]$:

Mean Value Theorem of Integral Calculus: If $f : [a, b] \to \mathbb{R}$ is continuous. Then there is a $\xi \in (a, b)$ with the property that

$$f(\xi) \cdot (b - a) = \int_a^b f(x)\, dx.$$



Proof: Due to its definition, the integral over a constant function $f(x) = c$ is

$$\int_a^b c\, dx = c \cdot (b - a)\,.$$

If $f(x)$ is non-constant, we set $m := \min\limits_{x \in [a, b]} f(x)$ as minimum and $M := \max\limits_{x \in [a, b]} f(x)$ as the maximum of the function f in the interval $[a, b]$. Then there is a $\underline{x}$ with $f(\underline{x}) = m$ and a $\bar{x}$ with $f(\bar{x}) = M$ such that

$$f(\underline{x}) \le f(x) \le f(\bar{x})\,.$$

Since $f(\underline{x})$ and $f(\bar{x})$ are constant numbers, the following applies

$$f(\underline{x})\,(b - a) = \int_a^b f(\underline{x})\, dx \le \int_a^b f(x)\, dx \le \int_a^b f(\bar{x})\, dx = f(\bar{x})\,(b - a)$$

$$\Rightarrow \quad f(\underline{x}) \le \frac{1}{b - a} \int_a^b f(x)\, dx \;\le\; f(\bar{x})\,.$$

According to the intermediate value theorem there is then again a $\xi \in (a, b)$ with the function value

$$f(\xi) = \frac{1}{b - a} \int_a^b f(x)\, dx.$$

In the above consideration, we used the monotonic property of the definite integral which states that with $g(x) \le f(x) \le h(x)$ it follows:

$$\int_a^b g(x)\, dx \le \int_a^b f(x)\, dx \le \int_a^b h(x)\, dx.$$

This monotonic property is directly checked on the basis of the algebraic definition of the integral. □

A more general formulation of the mean value theorem is:

Theorem: (General Mean Value Theorem of the Integral Calculus).
Let $f, \varphi : [a, b] \to \mathbb{R}$ be continuous functions and $\varphi \geq 0$. Then there is a point $\xi \in (a, b)$ with the property that

$$\int_a^b f(x)\, \varphi(x)\, dx = f(\xi) \int_a^b \varphi(x)\, dx.$$

We now establish the relation between differential and integral calculus. This relation is not only of theoretical importance, it also provides a practical method for the calculation of integrals.

Integral Function
Integralfunktion

Theorem of Integral Functions:
Is $f : [a, b] \to \mathbb{R}$ continuous and $F(x) := \int_a^x f(t)\, dt$ an integral function with respect to f. Then F is differentiable and it applies:

$$F'(x) = f(x).$$

Important: The theorem about the integral function states that the derivative of the integral function $F(x)$ gives the integrand $f(x)$!

Proof: We consider the difference of the areas

$$\Delta F = F(x+h) - F(x) = \int_a^{x+h} f(t)\, dt - \int_a^x f(t)\, dt = \int_x^{x+h} f(t)\, dt$$

and apply to the integral of the right side the mean theorem of the integral calculus: $\int_x^{x+h} f(t)\, dt = h\, f(\xi_h)$ with $\xi_h \in (x, x+h)$. Then we form the difference quotient

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\, \Delta F = f(\xi_h).$$

For $h \to 0$ the left side converges against $F'(x)$ and the right side against $f(x)$. Since $\xi_h \stackrel{h \to 0}{\longrightarrow} x$ and f is continuous, we have: $f(\xi_h) \stackrel{h \to 0}{\longrightarrow} f(x)$. □

Examples 8.4:

① For $f(x) = 1$, $F(x) = \int_0^x 1\, dt = x$, then $F'(x) = 1$.

② For $g(x) = x$, $G(x) = \int_0^x t\, dt = \frac{x^2}{2}$, then $G'(x) = x$.

③ For $h(x) = x^2$, $H(x) = \int_0^x t^2\, dt = \frac{x^3}{3}$, then $H'(x) = x^2$. □

Antiderivative functions

In general, we have shown that the derivative of the integral function $F(x) = \int_a^x f(t)\, dt$ returns the integrand $f(x)$ as the result. We call such functions $F(x)$ with $F'(x) = f(x)$ *antiderivative functions*:

Definition: *Every function $F(x)$ with $F'(x) = f(x)$ is called* **antiderivative function** *of $f(x)$.*

Antiderivative Function
Stammfunktion

With this usage it is possible to reformulate the theorem about the integral functions: **Any indefinite integral**

$$I(x) = \int_a^x f(t)\, dt$$

is an antiderivative function of $f(x)$, i.e. $I'(x) = f(x)$.

Example 8.5. In the following table, for some functions $f(x)$ an antiderivative function $F(x)$ is given. The property $F' = f$ can be directly recalculated.

$f(x)$	$x^n,\ n \neq -1$	x^{-1}	$\sqrt[n]{x}$	e^x	$\sin x$	$\cos x$
$F(x)$	$\dfrac{x^{n+1}}{n+1}$	$\ln x$	$\dfrac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}$	e^x	$-\cos x$	$\sin x$

Table 8.2: Elementary antiderivative functions □

Remark: For every continuous function there are infinitely many antiderivative functions, because e.g. for x^n both $\frac{1}{n+1}x^{n+1}$ and $\frac{1}{n+1}x^{n+1}+2$ as well as $\frac{1}{n+1}x^{n+1}+C$ is an antiderivative function. However, two antiderivative functions of a f function differ at most by a constant:

Theorem: If F_1 and F_2 are two antiderivative functions of f, they differ at most by a additive constant $C \in \mathbb{R}$:

$$F_1(x) = F_2(x) + C.$$

Proof: Since F_1 and F_2 are antiderivative functions to f, the following applies $F_1'(x) = f(x) = F_2'(x)$. Therefore $(F_1(x) - F_2(x))' = 0 \quad \Rightarrow$ $F_1(x) - F_2(x) = const.$ □

Consequently, any indefinite integral can be written in the form of

$$\int_a^x f(t)\,dt = F(x) + C,$$

where $F(x)$ is any antiderivative function and C is a constant. For every continuous function $f(x)$ there are infinitely many indefinite integrals. Therefore, this set of functions is characterized by the omission of the integration boundaries

$$\int f(x)\, dx = \{ \textit{ Set of all indefinite integrals of } \; f(x)\}.$$

Integration Constant
Integrationskonstante

Since all antiderivative functions differ only by one constant, we write **in short**

$$\int f(x)\, dx = F(x) + C,$$

and call C the *integration constant.*

Examples 8.6:

① $\displaystyle\int e^x\, dx = e^x + C.$

② $\displaystyle\int x^k\, dx = \tfrac{1}{k+1}\, x^{k+1} + C \qquad (k \neq -1)\,.$

③ $\displaystyle\int \cos x\, dx = \sin x + C.$

⚠ **Remark:** One reason why integral calculus is more difficult than differential calculus is that not every antiderivative function can be represented by elementary functions. The functions

$$f(x) = e^{x^2}, \quad f(x) = \frac{\sin x}{x}$$

have no elementary representable antiderivative functions!

Table of Antiderivative Functions. In the following table, the set of all antiderivative functions $\int f(x)\,dx = F(x)$ is compiled for many elementary functions. The validity can mostly be confirmed very easily with the relation $F'(x) = f(x)$.

Table 8.3: Antiderivative Functions

$f(x) = F'(x)$	Antiderivative function F : $F(x) = \int f(x)\,dx + C$	Domain $\mathbb{D}_f$
$k \ \ (k \in \mathbb{R})$	$k\,x + C$	$\mathbb{R}$
$x^{\alpha} \ \ (\alpha \neq -1)$	$\frac{1}{\alpha+1}\,x^{\alpha+1} + C$	$\mathbb{R}_{>0}$
x^{-1}	$\ln\|x\| + C$	$\mathbb{R}\setminus\{0\}$
$\sin x$	$-\cos x + C$	$\mathbb{R}$
$\cos x$	$\sin x + C$	$\mathbb{R}$
$\tan x$	$-\ln\|\cos x\| + C$	$\mathbb{R}\setminus\{x = \frac{\pi}{2} + k\,\pi,\ k \in \mathbb{Z}\}$
$\cot x$	$\ln\|\sin x\| + C$	$\mathbb{R}\setminus\{x = k\,\pi,\ k \in \mathbb{Z}\}$
$a^x \ \ (a > 0, \neq 1)$	$\frac{a^x}{\ln a} + C$	$\mathbb{R}$
e^x	$e^x + C$	$\mathbb{R}$
$e^{a\,x} \ \ (a \neq 0)$	$\frac{1}{a}\,e^{a\,x} + C$	$\mathbb{R}$
$\ln x$	$x \cdot \ln x - x + C$	$\mathbb{R}_{>0}$
$\frac{1}{\cos^2 x}$	$\tan x + C$	$\mathbb{R}\setminus\{x = \frac{\pi}{2} + k\,\pi,\ k \in \mathbb{Z}\}$
$\frac{1}{\sin^2 x}$	$-\cot x + C$	$\mathbb{R}\setminus\{x = k\,\pi,\ k \in \mathbb{Z}\}$
$\sin^2 x$	$\frac{1}{2}\,(x - \sin x \cdot \cos x) + C$	$\mathbb{R}$
$\cos^2 x$	$\frac{1}{2}\,(x + \sin x \cdot \cos x) + C$	$\mathbb{R}$
$\tan^2 x$	$\tan x - x + C$	$\mathbb{R}\setminus\{x = \frac{\pi}{2} + k\,\pi,\ k \in \mathbb{Z}\}$
$\cot^2 x$	$-\cot x - x + C$	$\mathbb{R}\setminus\{x = k\,\pi,\ k \in \mathbb{Z}\}$

$f(x) = F'(x)$	Antiderivative function F : $F(x) = \int f(x)\ dx + C$	Domain $\mathbb{D}_f$
$\arcsin x$	$x \cdot \arcsin x + \sqrt{1-x^2} + C$	$(-1,\ 1)$
$\arccos x$	$x \cdot \arccos x - \sqrt{1-x^2} + C$	$(-1,\ 1)$
$\arctan x$	$x \cdot \arctan x - \frac{1}{2}\ \ln\left(x^2+1\right) + C$	$\mathbb{R}$
$\operatorname{arccot} x$	$x \cdot \operatorname{arccot} x + \frac{1}{2}\ \ln\left(x^2+1\right) + C$	$\mathbb{R}$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin x + C$	$(-1,\ 1)$
$\frac{-1}{\sqrt{1-x^2}}$	$\arccos x + C$	$(-1,\ 1)$
$\frac{1}{1+x^2}$	$\arctan x + C$	$\mathbb{R}$
$\frac{-1}{1+x^2}$	$\operatorname{arccot} x + C$	$\mathbb{R}$

$f(x) = F'(x)$	Antiderivative function $F(x) = \int f(x)\ dx + C$	Domain $\mathbb{D}_f$
$\sinh x$	$\cosh x + C$	$\mathbb{R}$
$\cosh x$	$\sinh x + C$	$\mathbb{R}$
$\tanh x$	$\ln(\cosh x) + C$	$\mathbb{R}$
$\coth x$	$\ln\lvert\sinh x\rvert + C$	$\mathbb{R}\setminus\{0\}$
$\frac{1}{\cosh^2 x}$	$\tanh x + C$	$\mathbb{R}$
$\frac{1}{\sinh^2 x}$	$-\coth x + C$	$\mathbb{R}\setminus\{0\}$
$\frac{1}{\sqrt{1+x^2}}$	$ar\sinh x + C$	$\mathbb{R}$
$\frac{1}{\sqrt{x^2-1}}$	$ar\cosh x + C$	$(1,\ \infty)$
$\frac{1}{1-x^2}$	$ar\tanh x + C$	$(-1,\ 1)$
$\frac{1}{1-x^2}$	$ar\coth x + C$	$\mathbb{R}\setminus[-1,\ 1]$

However, antiderivative functions are neither geometrically nor physically as important as the definite integrals. So far we have calculated explicitly only $\int_0^b x^2\,dx = \frac{b^3}{3}$. By the following main theorem on differential and integral calculus the calculation of definite integrals is attributed to a simpler task, namely the search for antiderivative functions:

Fundamental Theorem of Calculus

Fundamentalsatz der Differential- und Integralrechnung

Fundamental Theorem of Calculus:
Let $f : [a, b] \to \mathbb{R}$ be a continuous function and F an antiderivative function of f. Then

$$\int\limits_a^b f(x)\,dx = F(b) - F(a)\,.$$

Proof: Let $x \in [a, b]$ and $F_0(x) := \int_a^x f(t)\,dt$. Then $F_0(x)$ a antiderivative function of f with $F_0(a) = 0$ and $F_0(b) = \int_a^b f(t)\,dt$. If $F(x)$ is an arbitrary antiderivative function of f: $F - F_0 = const = c$, then

$$\begin{aligned} F(b) - F(a) &= F_0(b) + c - (F_0(a) + c) \\ &= F_0(b) - F_0(a) = F_0(b) = \int_a^b f(t)\,dt. \end{aligned}$$ □

Thus, the calculation of definite integrals takes place in two steps:

Calculation of Definite Integrals:

(1) Determine an antiderivative function $F(x)$ for the integrand $f(x)$.

(2) With this antiderivative function the difference $F(b) - F(a)$ is calculated :

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(x)|_a^b = F(b) - F(a)\,.$$

Subsequently, $[F(x)]_a^b = F(x)|_a^b$ is used as an abbreviated notation for the difference $F(b) - F(a)$.

Examples 8.7 (Calculation of Definite Integrals):

① $\int_a^b x^3\,dx =$?: According to Table 8.3 an antiderivative function is $\frac{1}{4}\,x^4$, so

$$\int_a^b x^3\,dx = \frac{1}{4}\,x^4\Big|_a^b = \tfrac{1}{4}\,\left(b^4 - a^4\right).$$

For $0 \le a < b$ this is the area of the curve $y = x^3$ with the x-axis in the range of $a \le x \le b$.

② $\int_0^\pi \sin x\,dx =$?: An antiderivative function of $\sin(x)$ is $F(x) = -\cos(x)$, so

$$\int_0^\pi \sin x\,dx = -\cos x|_0^\pi = -\cos\pi - (-\cos(0)) = 1 - (-1) = 2.$$

This is the area under the sinusoid in the first half period.

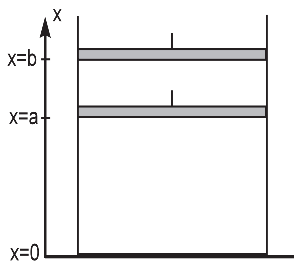
Application Example 8.8 (Expansion Work of Gases).

In a cylinder of the base area $F\ \left[cm^2\right]$ a gas is compressed by a movable piston. If the piston's height is $x\ [cm]$ from the cylinder bottom, the gas pressure in the cylinder is $p\,(x)\ \left[\frac{g}{cm\,s^2}\right]$. When shifting the piston from $x = a$ to $x = b$ the *work* of the gas is given by

$$A = \int_a^b F\,p\,(x)\ dx.$$

As a special case we consider the *isothermal expansion* of an ideal gas with the equation of state

$$p\,(x)\cdot V\,(x) = p\,(a)\cdot V\,(a) = const$$
(Boyle-Mariotte's law)



With volume $V\,(x) = F\cdot x$, we get

$$p\,(x) = \tfrac{p(a)\cdot V(a)}{V(x)} = \tfrac{p(a)\cdot V(a)}{F\cdot x}.$$

$$A = F\int_a^b \frac{p\,(a)\cdot V\,(a)}{F\cdot x}\,dx = p\,(a)\cdot V\,(a)\int_a^b \tfrac{1}{x}\,dx.$$

According to Table 8.3 we obtain

$$A = p\,(a)\cdot V\,(a)\cdot[\ln x]_a^b = p\,(a)\cdot V\,(a)\cdot[\ln(b) - \ln(a)]$$
$$= p\,(a)\cdot V\,(a)\cdot\ln\frac{b}{a}\,.$$

□

8.3 Rules of Integral Calculus

The calculation of definite integrals is simplified with the help of integration rules. They result directly from the definition of the definite integral as the limit value of the subtotal. All functions occurring are assumed to be continuous.

Constant Factor Rule

Faktorregel

Constant factor rule: A constant factor c may be placed before the integral:

$$\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx.$$

Example 8.9. $\int_0^{\pi/2} 4\cos x\, dx = 4 \int_0^{\pi/2} \cos x\, dx$

$$= 4\, [\sin x]_0^{\pi/2} = 4\left(\sin(\frac{\pi}{2}) - \sin(0)\right) = 4.$$ □

Sum Rule

Summenregel

Sum rule: A sum of functions may be integrated term by term:

$$\int_a^b (f_1(x) + f_2(x))\, dx = \int_a^b f_1(x)\, dx + \int_a^b f_2(x)\, dx.$$

Example 8.10. $\int_0^1 (-3x^2 + x)\, dx = -3 \int_0^1 x^2\, dx + \int_0^1 x\, dx$

$$= -3\left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}.$$ □

Remarks:

(1) The factor and sum rules also apply analogously to indefinite integrals.

(2) Previously, $a < b$ was always assumed. The factor and sum rules remain valid for any real numbers a, b from the domain of f, if we add the following definition:

Integration Boundaries
Integrationsgrenzen

Integration boundaries:

(1) *Integration boundaries coincide:* $\int_a^a f(x)\, dx = 0.$

(2) *Swap the integration bounds:* $\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx.$

Examples 8.11:

① $\int_2^2 \frac{1}{x}\, dx = \ln x|_2^2 = \ln(2) - \ln(2) = 0.$

② $\int_{\pi/2}^0 \sin x\, dx = -\int_0^{\pi/2} \sin x\, dx = -\left[-\cos x\right]_0^{\pi/2} = -1.$ □

Additivity of Integral
Additivität des Integral

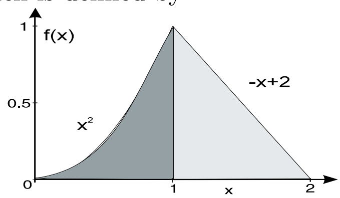
Additivity of the Integral: For any position c inside the integration range $a \le c \le b$ of f, the following applies

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

The additivity of the integral is exploited when a function has different function rules on partial intervals.

Example 8.12. Given is the function $f(x)$, which is defined by

$$f(x) = \begin{cases} x^2 & \text{for } 0 \le x \le 1 \\ -x+2 & \text{for } 1 \le x \le 2 \end{cases}$$



To calculate the area below the curve f in the range $[0, 2]$, the integral must be split into an integral over $[0, 1]$ and $[1, 2]$, since the function has different function rules in both sub-intervals. Therefore

$$A = \int_0^2 f(x)\, dx = \int_0^1 x^2\, dx + \int_1^2 (-x+2)\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{-x^2}{2} + 2x\right]_1^2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$ □

8.4 Integration Methods

In practical cases, the integration of functions often proves to be more difficult than differentiation. While differentiation can be achieved by applying simple rules (product rule, quotient rule, chain rule), integration is more difficult. Nevertheless, in many cases a antiderivative function can be found by one of the following integration methods.

8.4.1 Integration by Parts

The *Integration by Parts* is the counterpart to the product rule of differentiation, which states that

$$(u(x) \cdot v(x))' = u'(x)\ v(x) + u(x)\ v'(x)\,.$$

We resolve this equation with respect to $u(x)\ v'(x)$ and integrate

$$\begin{aligned} u(x)\ v'(x) &= (u(x)\ v(x))' - u'(x)\ v(x) \\ \int_a^b u(x)\ v'(x)\ dx &= \int_a^b (u(x)\ v(x))'\ dx - \int_a^b u'(x)\ v(x)\ dx. \end{aligned}$$

According to the fundamental theorem of differential and integral calculus

$$\int_a^b (u(x)\ v(x))'\ dx = [u(x) \cdot v(x)]_a^b\,,$$

so the following applies

Integration by Parts
Partielle Integration

Integration by Parts:

$$\int_a^b u(x)\ v'(x)\ dx = [u(x)\ v(x)]_a^b - \int_a^b u'(x)\ v(x)\ dx.$$

Remarks:

(1) Whether integration succeeds according to integration by parts depends on the "proper" (appropriate) choice of $u(x)$ and $v'(x)$.

(2) In some cases, the integration procedure must be applied several times before a basic integral is encountered.

(3) Especially when integrating functions which contain a trigonometric function as a factor, the case occurs that the integral to be calculated occurs again on the right side after one or more partial integration. In this case, the equation is resolved with respect to searched integral.

(4) The formula of integration by parts also applies to indefinite integrals

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx.$$

Examples 8.13 (Integration by Parts):

① Find $\int_1^2 x\, e^x\, dx$.
We set

$$\boxed{\begin{array}{lcl} u(x) = x & \Rightarrow & u'(x) = 1 \\ v'(x) = e^x & \Rightarrow & v(x) = e^x \end{array}}$$

and obtain

$$\int_1^2 x\, e^x\, dx = [x\, e^x]_1^2 - \int_1^2 1 \cdot e^x\, dx = [x\, e^x]_1^2 - [e^x]_1^2$$
$$= 2\, e^2 - e^1 - e^2 + e^1 = e^2.$$

Hence, $\int x\, e^x\, dx = e^x\,(x-1) + C$.

② Find $\int x^2 \cos x\, dx$.

We set

$$\boxed{\begin{array}{lcl} u(x) = x^2 & \Rightarrow & u'(x) = 2x \\ v'(x) = \cos x & \Rightarrow & v(x) = \sin x \end{array}}$$

and obtain

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x\, \sin x\, dx.$$

Re-partial integration of $\int 2x \sin x\, dx$ results in

$$\boxed{\begin{array}{lcl} u(x) = 2x & \Rightarrow & u'(x) = 2 \\ v'(x) = \sin x & \Rightarrow & v(x) = -\cos x \end{array}}$$

$$\int x^2 \cos x\, dx = x^2 \sin x - \left[2x\,(-\cos x) - \int 2\,(-\cos x)\, dx\right]$$

$$= x^2 \sin x + 2x\, \cos x - 2\, \sin x + C.$$ □

Tip: As a rule, $u(x)$ is set equal to the power factor in order to reduce this term by multiple integration by parts. But in some cases $v'(x) = 1 \Rightarrow v(x) = x$ leads to the desired result. In others, the integral to be calculated reappears on the right side of the equation. Then, the equation with respect to the integral is resolved:

③ Find $\int \ln x \, dx$.
With

$$\boxed{\begin{array}{lll} u(x) = \ln x & \Rightarrow & u'(x) = \dfrac{1}{x} \\ v'(x) = 1 & \Rightarrow & v(x) = x \end{array}}$$

results in

$$\begin{aligned} \int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx &= x \ln x - x + C \\ &= x \, (\ln x - 1) + C. \end{aligned}$$

④ Find $\int \cos^2 x \, dx = \int \cos x \cdot \cos x \, dx$.
With

$$\boxed{\begin{array}{lll} u(x) = \cos x & \Rightarrow & u'(x) = -\sin x \\ v'(x) = \cos x & \Rightarrow & v(x) = \sin x \end{array}}$$

is

$$\int \cos^2 x \, dx = \cos x \, \sin x - \int -\sin x \, \sin x \, dx = \cos x \, \sin x + \int \sin^2 x \, dx.$$

We replace $\sin^2 x = 1 - \cos^2 x$.

$$\Rightarrow \int \cos^2 x \, dx = \cos x \, \sin x + x - \int \cos^2 x \, dx.$$

We add $\int \cos^2 x \, dx$ on both sides and divide by the factor 2. Finally, we get

$$\int \cos^2 x \, dx = \tfrac{1}{2} \, (\sin x \, \cos x + x) + C.$$

□

8.4.2 Integration by Substitution

Besides integration by parts there is another very important integration technique: *integration by substitution*. This integration method can be derived starting with the chain rule. For the derivation of the function $G\,(f\,(x))$ with respect to x we obtain using chain rule:

$$\frac{d}{dx} G\,(f\,(x)) = G'\,(f\,(x)) \cdot f'\,(x)\,.$$

With $g(x) = G'(x)$ this results can be rewritten:

Integration by Substitution
Integration durch Substitution

Substitution Rule for Indefinite Integrals:

$$\int g\,(f\,(x))\; f'\,(x)\; dx = G\,(f\,(x)) + C,$$

if G is a antiderivative function of g.

Table 8.4 shows special cases of this general substitution rule. Note that cases (A), (B) and (C) are included as special cases in the general form (D).

Direct Substitutions
Explizite Substitutionen

	Integral type	Substitution	Solution
(A)	$\int g\,(a\,x+b)\;dx$	$y=a\,x+b$	$\frac{1}{a}\,G\,(a\,x+b)+C$
(B)	$\int f\,(x)\;f'\,(x)\;dx$	$y=f\,(x)$	$\frac{1}{2}\,f^2\,(x)+C$
(C)	$\int \frac{f'(x)}{f(x)}\,dx$	$y=f\,(x)$	$\ln\lvert f\,(x)\rvert+C$
(D)	$\int g\,(f\,(x))\;f'\,(x)\;dx$	$y=f\,(x)$	$G\,(f\,(x))+C$

Table 8.4: Integral substitutions

Examples 8.14 (Integral Substitutions According to Table 8.4):

(A1) $\int_2^3 (2x-3)^4\;dx = ?$

We first determine a antiderivative function and then use the upper and lower integration bounds to calculate the definite integral. To do this, we substitute $\boxed{y=2x-3}$ and replace each term of the integral that contains the integration variable x by a corresponding term with y. In particular, the differential dx must also be replaced by a corresponding term with dy.

From $y=2x-3 \quad \hookrightarrow y'=\frac{dy}{dx}=2 \quad \hookrightarrow dx=\frac{1}{2}\,dy$. Thus

$$\int (2x-3)^4\;dx = \int y^4\,\tfrac{1}{2}\,dy = \tfrac{1}{2}\int y^4\,dy = \tfrac{1}{2}\,\tfrac{1}{5}\,y^5 + C.$$

After calculation of the integral, y is replaced by back-substitution with $2x-3$:

$$\int (2x-3)^4\;dx = \tfrac{1}{10}\;(2x-3)^5 + C.$$

The particular integral is therefore

$$\int_2^3 (2x-3)^4\;dx = \left[\tfrac{1}{10}\;(2x-3)^5\right]_2^3 = \tfrac{1}{10}\;[243-1] = 24.2\,.$$

(A1) Alternatively, if the substitution method is carried out directly at the definite integral

$$\int_2^3 (2x-3)^4\;dx,$$

the integration bounds must also be replaced! After the calculation of the substituted integral there will be no longer a back-substitution. Starting from $y = 2x - 3$ follows for the lower limit $x_u = 2 \hookrightarrow y_u = 1$ and for the lower bound $x_o = 3 \hookrightarrow y_o = 3$:

$$\int_2^3 (2x-3)^4 \, dx = \int_1^3 y^4 \frac{1}{2} \, dy = \left[\tfrac{1}{10} \, y^5\right]_1^3 = \tfrac{1}{10} \, [243 - 1] = 24.2 \, .$$

(A2) $\displaystyle\int_0^1 \frac{1}{1+4x} \, dx = ?$

Substitute $\boxed{y = 1 + 4x}$ $\quad \hookrightarrow y' = \dfrac{dy}{dx} = 4 \quad \hookrightarrow dx = \tfrac{1}{4} \, dy.$

$$\hookrightarrow \int \frac{1}{1+4x} \, dx = \int \frac{1}{y} \cdot \frac{1}{4} \, dy = \frac{1}{4} \int \frac{1}{y} \, dy = \frac{1}{4} \, \ln |y| + C.$$

By back-substituting $y = 1 + 4x$

$$\int \frac{1}{1+4x} \, dx = \frac{1}{4} \, \ln |1 + 4x| + C.$$

Finally, the result is

$$\int_0^1 \frac{1}{1+4x} \, dx = \left[\frac{1}{4} \, \ln |1 + 4x|\right]_0^1 = \frac{1}{4} \, \ln 5 - \frac{1}{4} \, \ln 1 = \frac{1}{4} \, \ln 5 \, .$$

(B1) $\displaystyle\int \sin x \, \cos x \, dx = ?$

Substitute $\boxed{y = \sin x}$ $\quad \hookrightarrow y' = \dfrac{dy}{dx} = \cos x \quad \hookrightarrow dx = \dfrac{1}{\cos x} \, dy.$

$$\hookrightarrow \int \sin x \, \cos x \, dx = \int y \, \cos x \, \frac{1}{\cos x} \, dy = \int y \, dy = \frac{1}{2} \, y^2 + C.$$

By back-substituting $y = \sin x$

$$\int \sin x \, \cos x \, dx = \frac{1}{2} \, \sin^2 x + C.$$

(B2) $\int_1^2 \frac{\ln x}{x}\, dx = ?$

Substitution: $\boxed{y = \ln x}$ $\hookrightarrow y' = \frac{dy}{dx} = \frac{1}{x}$ $\hookrightarrow dx = x\, dy.$

$$\hookrightarrow \int \frac{\ln x}{x}\, dx = \int \frac{y}{x} \cdot x\, dy = \int y\, dy = \frac{1}{2}\, y^2 + C.$$

Back-substitution: $\int \frac{\ln x}{x}\, dx = \frac{1}{2}\, \ln^2 x + C.$

Calculation of the specific integral:

$$\int_1^2 \frac{\ln x}{x}\, dx = \frac{1}{2}\, \left[\ln^2 x\right]_1^2 = \frac{1}{2}\, \ln^2 2\,.$$

(C1) $\int \frac{2x-3}{x^2-3x+1}\, dx = \int \frac{2x-3}{y} \cdot \frac{dy}{2x-3}$

$$= \int \frac{1}{y}\, dy = \ln|y| + C = \ln\left|x^2 - 3x + 1\right| + C$$

with the substitution $\boxed{y = x^2 - 3x + 1}$ and $dx = \frac{1}{2x-3}\, dy.$

(C2) $\int_0^1 \frac{e^x}{2\, e^x + 5}\, dx = ?$

Substitution: $\boxed{y = 2\, e^x + 5}$ $\hookrightarrow y' = \frac{dy}{dx} = 2\, e^x$ $\hookrightarrow dx = \frac{1}{2\, e^x}\, dy.$

Upper bound $x_o = 1 \hookrightarrow y_o = 2\, e + 5$

Lower bound $x_u = 0 \hookrightarrow y_u = 7.$

$$\int_0^1 \frac{e^x}{2\, e^x + 5}\, dx = \int_7^{2e+5} \frac{e^x}{y} \cdot \frac{1}{2\, e^x}\, dy = \tfrac{1}{2} \int_7^{2e+5} \frac{1}{y}\, dy$$

$$= \tfrac{1}{2}\, \ln y\, |_7^{2e+5} = \tfrac{1}{2}\, [\ln(2\, e + 5) - \ln 7] = 0.1997\,.$$

(D1) $\int \left(x^3 + 2\right)^{\frac{1}{2}}\, x^2\, dx = ?$

Substitution: $\boxed{y = x^3 + 2}$ $\hookrightarrow \frac{dy}{dx} = 3x^2$ $\hookrightarrow dx = \frac{1}{3x^2}\, dy.$

$$\int \left(x^3 + 2\right)^{\frac{1}{2}}\, x^2\, dx = \int y^{\frac{1}{2}} \cdot x^2 \cdot \frac{1}{3x^2}\, dy = \frac{1}{3} \int y^{\frac{1}{2}}\, dy = \frac{1}{3}\, \frac{2}{3}\, y^{\frac{3}{2}} + C.$$

Back-substitution:

$$\int (x^3+2)^{\frac{1}{2}}\ x^2\,dx = \frac{2}{9}\ (x^3+2)^{\frac{3}{2}} + C.$$

(D2) $\displaystyle\int \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx = ?$

Substitution: $\boxed{y = x\,e^x}$ $\hookrightarrow \dfrac{dy}{dx} = e^x + x\,e^x$ $\hookrightarrow dx = \dfrac{1}{e^x + x\,e^x}\,dy.$

$$\int \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx = \int \frac{e^x + x\,e^x}{y^3}\cdot\frac{1}{e^x + x\,e^x}\,dy = \int y^{-3}\,dy = -\frac{1}{2}\,y^{-2} + C.$$

Back-substitution: $\displaystyle\int \frac{e^x + x\,e^x}{(x\,e^x)^3}\,dx = -\frac{1}{2}\ (x\,e^x)^{-2} + C.$ □

Tip: As a rule, a function $f(x)$ is searched for in the integrand, whose derivative $f'(x)$ occurs as factor. Then the function $y = f(x)$ is substituted!

Tip: If a substitution is performed for a *definite* integral, the integration bounds must also be replaced.

Note: Besides the substitution rules given in Table 8.4 there are many others. Some of them are described in Table 8.5:

Implicit Substitutions

Implizite Substitutionen

Table 8.5: Implicit Integral Substitutions

	Integral type	Substitution
(E)	$\int g\left(x,\ \sqrt{a^2 - x^2}\right)\ dx$	$x = a \cdot \sin(y)$
(F)	$\int g\left(x,\ \sqrt{x^2 + a^2}\right)\ dx$	$x = a \cdot \sinh(y)$
(G)	$\int g\left(x,\ \sqrt{x^2 - a^2}\right)\ dx$	$x = a \cdot \cosh(y)$

Examples 8.15 (Integral Substitutions according to Table 8.5):

(E1) $\displaystyle\int \frac{dx}{\sqrt{4 - x^2}} = \int \frac{2\,\cos(y)}{2\,\cos(y)}\,dy = \int dy = y + C = \arcsin\left(\tfrac{1}{2}x\right) + C$

with the substitution $\boxed{x = 2\,\sin(y)}$ $\hookrightarrow \dfrac{dx}{dy} = 2\,\cos(y) \hookrightarrow dx = 2\,\cos(y)\ dy$

and $\sqrt{4-x^2} = \sqrt{4-4\sin^2(y)} = 2\cos(y)$, as $\cos^2(y) + \sin^2(y) = 1$.

$$\text{(E2)} \quad \int \frac{x}{\sqrt{4-x^2}}dx = \int \frac{2\sin(y)\,2\cos(y)}{2\cos(y)}\,dy = 2\int \sin(y)\,dy$$

$$= -2\cos(y) + C = -2\sqrt{1-\sin^2(y)} + C$$

$$= -2\sqrt{1-\frac{x^2}{4}} + C == -\sqrt{4-x^2} + C$$

with the substitution: $\boxed{x = 2\sin(y)}$
und $y = \arcsin(\frac{x}{2})$.

$$\text{(F)} \quad \int \frac{dx}{\sqrt{1+x^2}} = \int \frac{\cosh(y)}{\cosh(y)}\,dy = \int dy =$$

$$y + C = ar\sinh(x) + C = \ln\left(x + \sqrt{1+x^2}\right) + C$$

with the substitution $\boxed{x = \sinh(y)}$ $\hookrightarrow \dfrac{dx}{dy} = \cosh(y) \hookrightarrow dx = \cosh(y)\;dy$

and $\sqrt{1+x^2} = \sqrt{1+\sinh^2(y)} = \cosh(y)$, since $\cosh^2(y) - \sinh^2(y) = 1$.

$$\text{(G)} \quad \int \frac{dx}{\sqrt{x^2-25}} = \int \frac{5\sinh(y)}{5\sinh(y)}\,dy = \int dy =$$

$$y + C = ar\cosh\left(\frac{x}{5}\right) + C = \ln\left(\frac{x}{5} + \sqrt{\left(\frac{x}{5}\right)^2 + 1}\right) + C$$

with the substitution $\boxed{x = 5\cosh(y)}$$\hookrightarrow \dfrac{dx}{dy} = 5\sinh(y) \hookrightarrow dx = 5\sinh(y)\,dy$

and $\sqrt{x^2-25} = \sqrt{25\cosh^2(y) - 25} = 5\sqrt{\cosh^2(y) - 1} = 5\sinh(y)$,

as $\cosh^2(y) - \sinh^2(y) = 1$. □

Despite the variety of substitutions, there are no general formulas for the calculation of integrals that always lead to a result!

8.4.3 Partial Fraction Decomposition

For rational functions $f(x) = \frac{Z(x)}{N(x)}$ $(Z(x), N(x)$ Polynomials) there is a special integration technique, the so-called *partial fractional decomposition*. By this method rational functions can be integrated in closed form. For the execution they must be available in a proper rational representation. A rational function is called *proper rational* if the degree of the upper polynomial is smaller than the degree of the lower polynomial, otherwise the function is called *improper rational*. In case of a improper rational function, *polynomial division* is used to ensure that the degree of the numerator is then smaller than the degree of the denominator:

Example 8.16. $\dfrac{2x^3 - 2x^2 - 5x + 7}{x^2 - 3x + 2}$:

$$\begin{array}{rrrrl} (2x^3 & -2x^2 & -5x & +7) & : (x^2 - 3x + 2) = 2x + 4 + \dfrac{3x-1}{x^2-3x+2} \\ -(2x^3 & -6x^2 & +4x) & & \\ \hline & 4x^2 & -9x & +7 & \\ & -(4x^2 & -12x & +8) & \\ \hline & & 3x & -1 & \end{array}$$

□

A proper rational function can clearly be broken down into partial fractions. In the following we always assume that

$$\boxed{f(x) = \frac{p(x)}{q(x)}}$$

is a proper rational function with degree $(p) <$ degree $(q) = n$.

Partial Fraction Decomposition

Partialbruch-Zerlegung

Theorem: In case that $q(x) = a_n\,(x - x_1)\,(x - x_2) \cdot \ldots \cdot (x - x_n)$ has n **simple real zeros**, then the rational $f(x)$ can be decomposed in form of

$$f(x) = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \ldots + \frac{A_n}{x - x_n}$$

with **partial fractions** $\dfrac{A_i}{x - x_i}$.

Example 8.17. $\displaystyle\int \frac{3x - 1}{x^2 - 3x + 2}\, dx = ?$

In order to perform a partial fractional decomposition, the zeros of the denominator must first be determined. From

$$x^2 - 3x + 2 = 0$$

follows the pq formula $x_{1/2} = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}$. Therefore, $x_1 = 1$ and $x_2 = 2$ are the zeros of the denominator polynomial and $f(x) = \frac{3x-1}{x^2-3x+2} = \frac{3x-1}{(x-1)\,(x-2)}$. By the decomposition

$$f(x) = \frac{3x-1}{(x-1)\,(x-2)} = \frac{A_1}{x-1} + \frac{A_2}{x-2}$$

we get the partial decomposition. To calculate A_1 and A_2, the common denominator is formed as follows

$$\frac{A_1}{x-1} + \frac{A_2}{x-2} = \frac{A_1\,(x-2) + A_2\,(x-1)}{(x-1)\,(x-2)} \overset{!}{=} \frac{3x-1}{x^2-3x+2}.$$

We compare the top to $3x-1$:

$$A_1\,(x-2) + A_2\,(x-1) = 3x-1 \quad \text{for all} \;\; x.$$

To determine the constants A_1 and A_2, either a coefficient comparison is performed or special values for x are used:

$$\begin{aligned} x = 1: &\quad A_1\,(1-2) &= 3-1 &\quad \boxed{\Rightarrow A_1 = -2} \\ x = 2: &\quad A_2\,(2-1) &= 5 &\quad \boxed{\Rightarrow A_2 = 5.} \end{aligned}$$

Hence,

$$f(x) = \frac{-2}{x-1} + \frac{5}{x-2}$$

and we finally can integrate the partial decomposition instead of the original formula of the function

$$\begin{aligned} \int f(x)\,dx &= -2\int \frac{1}{x-1}\,dx + 5\int \frac{1}{x-2}\,dx \\ &= -2\,\ln|x-1| + 5\,\ln|x-2| + C \\ &= \ln|x-1|^{-2} + \ln|x-2|^5 + C = \ln\left|\frac{(x-2)^5}{(x-1)^2}\right| + C. \end{aligned}$$ □

If the denominator polynomial has double or multiple zeros, then the partial fractional decomposition must be modified for these zeros:

Theorem: $q(x)$ has **multiple real zeros**. Let x_l be a k-order zero, i.e. beside other zeroes the term $(x - x_l)$ with the power k occurs in the product representation of $q(x)$. Then this k-order zero is to be considered next to the others as follows:

$$f(x) = \ldots + \frac{B_1}{x-x_l} + \frac{B_2}{(x-x_l)^2} + \ldots + \frac{B_k}{(x-x_l)^k}.$$

Example 8.18. $\int \frac{2x^2+3x+1}{x^3-5x^2+8x-4}\,dx = ?$

$x = 1$ is a first order zero and $x = 2$ a second order zero of the denominator polynomial, since $x^3 - 5x^2 + 8x - 4 = (x-1)\,(x-2)^2$. For the partial fractional decomposition of the integrand f we therefore choose the approach

$$f(x) = \frac{A}{x-1} + \frac{B_1}{x-2} + \frac{B_2}{(x-2)^2}$$
$$= \frac{A\,(x-2)^2 + B_1\,(x-2)\,(x-1) + B_2\,(x-1)}{(x-1)\,(x-2)^2}\,.$$

After multiplication with the common denominator we get

$$2x^2 + 3x + 1 \stackrel{!}{=} A\,(x-2)^2 + B_1\,(x-2)\,(x-1) + B_2\,(x-1)\,.$$

We use special x values to determine A, B_1 and B_2:

$x = 1 : \boxed{6 = A}$

$x = 2 : \boxed{15 = B_2}$

$x = 0 : 1 = 4\,A + 2\,B_1 - B_2 \Rightarrow 1 = 9 + 2\,B_1 \quad \Rightarrow \boxed{B_1 = -4}$

Therefore

$$\int \frac{2x^2+3x+1}{x^3-5x^2+8x-4}\,dx$$
$$= 6\int \frac{1}{x-1}\,dx - 4\int \frac{1}{x-2}\,dx + 15\int \frac{1}{(x-2)^2}\,dx$$
$$= 6\,\ln|x-1| - 4\ln|x-2| - 15\,\frac{1}{x-2} + C.$$

□

Example 8.19. $\int \frac{x^6-2x^5+x^4+4x+1}{x^4-2x^3+2x-1}\,dx = ?$

(i) By polynomial division we decompose the integrand into a polynomial and a proper rational function:

$$\frac{x^6-2x^5+x^4+4x+1}{x^4-2x^3+2x-1} = \left(x^6-2x^5+x^4+4x+1\right) : \left(x^4-2x^3+2x-1\right)$$
$$= x^2 + 1 + \frac{x^2+2x+2}{x^4-2x^3+2x-1}$$

(ii) Since the denominator polynomial is of degree 4, the zeros of the denominator polynomial cannot be calculated by a formula. So it is necessary to guess a zero x_0 and then reduce the degree either by dividing the denominator polynomial by $(x - x_0)$ or by using Horner's scheme. For the denominator polynomial x^4-2x^3+2x-1 we get: $x_1 = 1$ is a triple order zero
$x_2 = -1$ is a first order zero

This gives the linear factorization

$$x^4 - 2x^3 + 2x - 1 = (x-1)^3\,(x+1)$$

(iii) Partial fractions are assigned to each zero point according to their multiplicity:

$$x_1 = 1: \quad \frac{A_1}{x-1} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3}\,;$$
$$x_2 = -1: \quad \frac{B}{x+1}\,.$$

Representation of $\frac{p(x)}{q(x)}$ by partial fractions:

$$\frac{x^2+2x+2}{x^4-2x^3+2x-1} = \frac{A_1}{(x-1)} + \frac{A_2}{(x-1)^2} + \frac{A_3}{(x-1)^3} + \frac{B}{x+1}$$

$$= \frac{A_1\,(x-1)^2\,(x+1) + A_2\,(x-1)\,(x+1) + A_3\,(x+1) + B\,(x-1)^3}{CD}$$

(iv) The coefficients are determined by multiplying by the common denominator (CD) and using special x values:

$$\begin{array}{lcll} x=1: & 5 & = A_3\cdot 2 & \Rightarrow \quad A_3 = \frac{5}{2} \\ x=-1: & 1 & = B\cdot(-2)^3 & \Rightarrow \quad B = -\frac{1}{8} \\ x=0: & 2 & = A_1 - A_2 + \frac{5}{2} + \left(-\frac{1}{8}\right)(-1) & \Rightarrow \quad A_1 - A_2 = -\frac{5}{8} \\ x=2: & 10 & = 3\,A_1 + 3\,A_2 + \frac{5}{2}\cdot 3 + \left(-\frac{1}{8}\right) & \Rightarrow \quad A_1 + A_2 = \frac{7}{8} \end{array}$$

By addition or subtraction of the last two equations we get

$$A_1 = \tfrac{1}{8}\,,\ A_2 = \tfrac{3}{4}\,.$$

(v) Finally, the integration of the polynomial and the partial fractions is performed:

$$\int \frac{x^6-2x^5+x^4+4x+1}{x^4-2x^3+2x-1}\,dx =$$
$$= \int \left(x^2+1\right)\,dx + \tfrac{1}{8}\int \frac{1}{x-1}\,dx + \tfrac{3}{4}\int \frac{1}{(x-1)^2}\,dx$$
$$+\tfrac{5}{2}\int \frac{1}{(x-1)^3}\,dx - \tfrac{1}{8}\int \frac{1}{x+1}\,dx =$$
$$= \tfrac{1}{3}\,x^3 + x + \tfrac{1}{8}\,\ln|x-1| - \tfrac{3}{4}\,\frac{1}{(x-1)} - \tfrac{5}{4}\,\frac{1}{(x-1)^2} - \tfrac{1}{8}\,\ln|x+1| + C.\square$$

Summary: Any rational function $f(x) = \dfrac{Z(x)}{N(x)}$ ($Z(x)$, $N(x)$ polynomials) can be integrated using algebraic methods if $N(x)$ decays into linear factors:

(1) Decomposition of the function $f(x) = r(x) + \dfrac{p(x)}{q(x)}$ into a polynomial $r(x)$ and a proper fractional function $\frac{p(x)}{q(x)}$.
(Note: $N(x) = q(x)$).

(2) Determination of the real zeros of $q(x)$ and their multiplicity.

(3) Each zero x_0 of $q(x)$ is assigned partial fractions according to its multiplicity.

$$x_0 \text{ first order zero} \quad \rightarrow \quad \frac{A}{x - x_0}$$

$$x_0 \text{ second order zero} \quad \rightarrow \quad \frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2}$$

$$\vdots$$

$$x_0 \ k\text{-order zero} \quad \rightarrow \quad \frac{A_1}{x - x_0} + \frac{A_2}{(x - x_0)^2} + \ldots + \frac{A_k}{(x - x_0)^k}.$$

The proper rational function $\frac{p(x)}{q(x)}$ can then be represented as the sum of all partial fractions.

(4) Determination of the constants occurring in the partial fractions (either by comparing coefficients and solving the corresponding linear equation system or by using special x values).

(5) Integration of $r(x)$ and all partial fractions with

$$\int \frac{1}{x - x_0}\, dx = \ln|x - x_0| + C;$$

$$\int \frac{1}{(x - x_0)^k}\, dx = -\frac{1}{k-1} \frac{1}{(x - x_0)^{k-1}} + C.$$

Amendments: According to the addition to the fundamental theorem of algebra (see Volume 1, Chapter 5) a real polynomial has exactly n zeros, which occur either real or in pairs in complex conjugations. For **complex zeros** the partial fractions apply:

(1) If the polynomial $q(x)$ in $x_0 = a + i\,b$ has a complex zero, then $\bar{x}_0 = a - i\,b$ is also a zero and the product

$$(x - x_0)\,(x - \bar{x}_0) = (x - a)^2 + b^2$$

cannot be decomposed. All such simple complex zeros are to be considered in the approach beside the remaining zeros by

$$f(x) = \ldots + \frac{C\,x + D}{(x-a)^2 + b^2} + \ldots$$

(2) If the complex zeros are k-order, the approach is modified

$$f(x) = \ldots + \frac{C_1\,x + D_1}{(x-a)^2 + b^2} + \frac{C_2\,x + D_2}{\left[(x-a)^2 + b^2\right]^2} + \ldots + \frac{C_k\,x + D_k}{\left[(x-a)^2 + b^2\right]^k} + \ldots$$

Examples 8.20 (With Maple-Worksheet):

① $\displaystyle\int \frac{2x^3 + x^2 + 2x + 2}{x^4 + 2x^2 + 1}\,dx = ?$: The zeros of the denominator $q(x)$ are $-i$, $-i$, i, i. I.e. $x = i$ and $x = -i$ are double zeros. The decomposition of the integrand into partial fractions is as follows

$$\frac{2x}{x^2+1} + \frac{1}{x^2+1} + \frac{1}{(x^2+1)^2}$$

and the subsequent integration results in

$$\ln\left(x^2+1\right) + \frac{3}{2}\arctan(x) + \frac{x}{2\,(x^2+1)} + C$$

② $\displaystyle\int \frac{x^3 + 2x^2 - 1}{x^4 - 2x^3 + 2x^2 - 2x + 1}\,dx = ?$: The zeros of the denominator are 1 (double) and $\pm i$. A partial fractional decomposition results in

$$\frac{1-3x}{2\,(x^2+1)} + \frac{5}{2\,(x-1)} + \frac{1}{(x-1)^2}$$

with the subsequent integration

$$-\frac{3}{4}\ln\left(x^2+1\right) + \frac{1}{2}\arctan(x) + \frac{5}{2}\ln(x-1) - \frac{1}{x-1} + C. \qquad \square$$

8.5 Improper Integrals

Until now we assumed that the integration bounds were finite. But it also happens in applications that they are not limited and the integrals nevertheless exist. These integrals are considered as special case of the already introduced definite integral.

Improper Integrals

Uneigentliche Integrale

Definition: *Integrals, where the integration limits are* $\pm\infty$, *are called* **Improper Integrals:**

$$\int_a^\infty f(x)\,dx\,,\quad \int_{-\infty}^b f(x)\,dx\,,\quad \int_{-\infty}^\infty f(x)\,dx.$$

Application Example 8.21 **(Work function).**

In the gravitational field of the earth a mass m shall be brought from the distance r_0 into infinity ($r = \infty$). Which work W_∞ has to be done and which velocity (= *escape velocity*) does the mass need? The work W_R, that has to be done to get the mass from $r = r_0$ to $r = R$, is based on the law of gravity

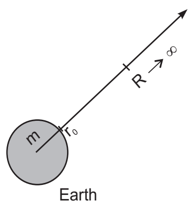


Figure 8.2. Work

$$F(r) = G\,\frac{mM}{r^2}$$

given by

$$W_R = \int_{r_0}^R F(r)\,dr = \int_{r_0}^R G\,\frac{m\,M}{r^2}\,dr = G\,m\,M\int_{r_0}^R \frac{1}{r^2}\,dr = G\,m\,M\left[-\frac{1}{r}\right]_{r_0}^R$$

$$= G\,m\,M\left(\frac{1}{r_0} - \frac{1}{R}\right).$$

G is the gravitational constant and M the mass of earth. For $R \to \infty$ we conclude

$$W_\infty = \lim_{R\to\infty} W_R = \lim_{R\to\infty} G\,m\,M\left[\frac{1}{r_0} - \frac{1}{R}\right] = \frac{G\,m\,M}{r_0}.$$

This amount of work equals the kinetic energy $\frac{1}{2}\,m\,v^2$, which the mass must have at the beginning; so the escape velocity is

$$\boxed{v_\infty = \sqrt{\frac{2\,G\,M}{r_0}}\,.}$$

□

The procedure chosen in the above example, i.e. first integrating r_0 to R and then letting R go to ∞, is the calculation method of improper integrals:

Calculation of Improper Integrals $\int_a^\infty f(x)\, dx$:

(1) Determination of the integral function $I(\lambda)$ as a function of the upper bound $I(\lambda) = \int_a^\lambda f(x)\, dx$.

(2) Evaluation of the limit value of the integral function for $\lambda \to \infty$:

$$\int_a^\infty f(x)\, dx = \lim_{\lambda\to\infty} I(\lambda) = \lim_{\lambda\to\infty} \int_a^\lambda f(x)\, dx.$$

Examples 8.22:

① $\int_1^\infty \frac{1}{x^3}\, dx = ?$

$$\int_1^\infty \frac{1}{x^3}\, dx = \lim_{\lambda\to\infty} \int_1^\lambda \frac{1}{x^3}\, dx = \lim_{\lambda\to\infty} \left[-\frac{1}{2}\frac{1}{x^2}\right]_1^\lambda = \lim_{\lambda\to\infty} \frac{1}{2}\left(-\frac{1}{\lambda^2}+1\right) = \frac{1}{2}.$$

② $\int_1^\infty \frac{1}{r}\, dr = ?$ This improper integral does **not** exist:

$$\int_1^\infty \frac{1}{r}\, dr = \lim_{\lambda\to\infty} \int_1^\lambda \frac{1}{r}\, dr = \lim_{\lambda\to\infty} \ln r|_1^\lambda = \lim_{\lambda\to\infty} \ln(\lambda) = \infty\,. \qquad \square$$

Application Example 8.23 **(RL Circuit).**

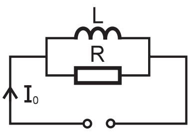


Figure 8.3. RL circuit

A coil (inductance L) and an ohmic resistor R are connected in parallel. A constant current I_0 flows. At the time $t_0 = 0$ the current source is switched off and the current decreases according to $I(t) = I_0\, e^{-\frac{R}{L}t}$. The energy available in form of a magnetic field is given by

$$E = \int_0^\infty R\, I^2(t)\, dt = \lim_{T\to\infty} \int_0^T R\, I_0^2\, e^{-2\frac{R}{L}t}\, dt$$

$$= R\, I_0^2 \lim_{T\to\infty} \left[-\frac{L}{2\,R}\, e^{-2\frac{R}{L}t}\right]_0^T = \frac{1}{2}\, L\, I_0^2\,. \qquad \square$$

Remarks:

(1) Important *transformations* for applications, the *Fourier Transform* and *Laplace Transform*, are defined by improper integrals.

(2) Integrals with *unrestricted* integrands are also called **improper integrals**:

$$\int_0^1 \frac{1}{\sqrt{1-t}}\,dt$$

is such an improper integral, since the integrand $\frac{1}{\sqrt{1-t}}$ is defined only for $0 \le t < 1$. Nevertheless, the integral has a finite value, since

$$\int_0^T \frac{1}{\sqrt{1-t}}\,dt = -2\sqrt{1-t}\Big|_0^T = 2 - 2\sqrt{1-T} \quad \overset{T\to 1}{\longrightarrow} \quad 2\,.$$

(3) A distinction is therefore made between three forms of improper integrals:
1. The integration interval is unlimited.
2. The integrand is unlimited.
3. Both the integration interval and the integrand are unlimited.

8.6 Applications of Integral Calculus

8.6

In this section some important applications of integral calculus are given. Further applications of integral calculus like the averaging property, the arc length and the curvature behavior as well as the calculation of rotational bodies together with the visualization in MAPLE can be found on the homepage.

8.6.1 Area calculations

Area Calculations
Flächenberechnung

Due to its definition, the integral is initially used to calculate areas. A surface is limited by $x = a$, $x = b$, by the x-axis and the function $f(x)$. Then the value of the area is given by

$$\int_a^b f(x)\,dx.$$

Application Example 8.24 :

We are looking for the area beneath a sinusoidal half-wave (see Fig. 8.4):

$$A = \int_0^\pi \sin x\,dx = -\cos x|_0^\pi = -(-1-1) = 2\,.$$

The area beneath the curve is 2. □

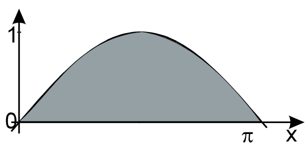


Figure 8.4. Area under function

The **area between two curves** $y = f(x)$ and $y = g(x)$ is calculated from the difference between the individual integrals:

$$A = \int_a^b (f(x) - g(x))\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx.$$

Application Example 8.25 **(Area between two curves).**

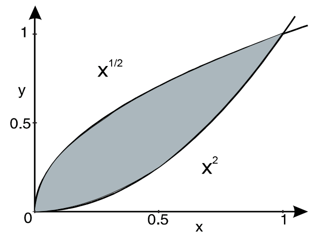


Figure 8.5.
Area between functions

The grey area between the function $y = \sqrt{x}$ and $y = x^2$ is to be found (see Fig. 8.5).

In order to calculate the grey hatched area, the intersection points of the curves must first be determined, since these provide the integration limits:

$$\sqrt{x} = x^2 \quad \hookrightarrow x = 0 \text{ and } x = 1.$$

The area enclosed by the curves is now calculated with the given integral over the difference of the functions:

$$A = \int_0^1 \left(\sqrt{x} - x^2\right) dx = \int_0^1 x^{\frac{1}{2}}\, dx - \int_0^1 x^2\, dx = \frac{2}{3}\, x^{\frac{3}{2}}\Big|_0^1 - \frac{1}{3}\, x^3\Big|_0^1 = \frac{1}{3}. \quad \square$$

8.6.2 Kinematics

The following applies to the movement of a mass point

$$v(t) = \dot{s}(t) = \tfrac{d}{dt}\, s(t) \qquad \text{(velocity)},$$

$$a(t) = \tfrac{d}{dt}\, v(t) = \dot{v}(t) = \ddot{s}(t) \qquad \text{(acceleration)}.$$

If the acceleration is known as a function of time (e.g. by a law of force), the velocity $v(t)$ follows by integration and the space-time-law $s(t)$ follows by a repeated integration:

Kinematics
Kinematik

$$v(t) = \int a(t)\, dt,$$

$$s(t) = \int v(t)\, dt.$$

Application Example 8.26 **(Free Fall without Air Friction).** For free fall without air resistance, the acceleration force is

$$m \cdot a = F_G = m\,g \quad \Rightarrow a\,(t) = g = const.$$

Thus the velocity is

$$v\,(t) = \int a(t)\,dt = g\,t + C_1.$$

The integration constant is determined by the initial velocity

$$v\,(0) = v_0 \quad \hookrightarrow \quad C_1 = v_0 \quad \Rightarrow \quad v\,(t) = g\,t + v_0.$$

The space-time-law follows through repeated integration

$$s\,(t) = \int v\,(t)\;dt = \int (g\,t + v_0)\;dt = \frac{g}{2}\,t^2 + v_0\,t + C_2.$$

The integration constant is determined from the initial position

$$s\,(0) = s_0 \quad \hookrightarrow \quad C_2 = s_0 \quad \Rightarrow \quad s\,(t) = \frac{1}{2}\,g\,t^2 + v_0\,t + s_0\,. \qquad \square$$

Application Example 8.27 **(Equation of Motion of a Rocket).** A rocket ascends vertically into the air and possesses a constant thrust force F_0. The mass of the rocket decreases due to the combustion of the fuel linearly, i.e.

$$m\,(t) = m_0 - q\,t = m_0\;(1 - \alpha\,t) \qquad \text{with} \;\; \alpha = \frac{q}{m_0}\,,$$

if m_0 the initial mass and q the fuel consumption. Assuming a constant acceleration due to gravity of g and without air resistance, the acceleration force or acceleration is

$$m\,a = F_0 - m\,g \quad \hookrightarrow \quad a = \frac{F_0}{m_0\;(1 - \alpha\,t)} - g\,.$$

The velocity is

$$v\,(t) = \int a\,(t)\;dt = \frac{F_0}{m_0} \int \frac{dt}{1 - \alpha\,t} - g \int dt = -\frac{F_0}{m_0}\,\frac{1}{\alpha}\,\ln\,(1 - \alpha\,t) - g\,t + C\,.$$

Note that the first integral is calculated with the substitution $y = 1 - \alpha\,t$. With the initial velocity $v\,(0) = 0$ $C = 0$ is calculated.

The space-time-law is obtained by repeated integration

$$s\,(t) = \int v\,(t)\;dt = -\frac{F_0}{m_0\,\alpha} \int \ln\,(1 - \alpha\,t)\;dt - g \int t\,dt\,.$$

With the substitution $y = 1 - \alpha\, t$ and the result from Example 8.13 ③ $\int \ln x = x \cdot (\ln x - 1) + C$ we get

$$s\,(t) = \frac{F_0}{m_0\,\alpha}\,\frac{1}{\alpha}\,\left[(1-\alpha\,t)\,\ln\,(1-\alpha\,t) - (1-\alpha\,t)\right] - \tfrac{1}{2}\,g\,t^2 + C\,.$$

With the initial condition $s\,(0) = 0$ follows $C = \frac{F_0}{m_0\,\alpha^2}$ and thus

$$s\,(t) = \frac{F_0}{m_0\,\alpha^2}\,\left[(1-\alpha\,t)\,\ln\,(1-\alpha\,t) + \alpha\,t\right] - \frac{1}{2}\,g\,t^2.$$ □

8.6.3 Electrodynamics

A point charge Q induces a radial **electric field** according to the formula

$$E\,(r) = \frac{1}{4\pi\,\varepsilon_0}\,\frac{Q}{r^2}$$

with the dielectric constant $\varepsilon_0 = 8.8542 \cdot 10^{-12}\,\frac{F}{m}$. The *voltage* U_{12} between two points P_1 and P_2 with distances r_1 and r_2 from Q is given by

Voltage Integral
Spannungsintegral

$$U_{12} = \int_{r_1}^{r_2} E\,(r)\;dr = \frac{Q}{4\pi\,\varepsilon_0}\int_{r_1}^{r_2}\frac{1}{r^2}\,dr = -\frac{Q}{4\pi\,\varepsilon_0}\,\frac{1}{r}\bigg|_{r_1}^{r_2} = \frac{Q}{4\pi\,\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

8.6.4 Energy Integrals

If a **location-independent** force F acts on a mass point m in path direction, then the executed *work* is by definition

$$W := F \cdot \Delta\,s,$$

if $\Delta\,s = s_E - s_A$ is the distance by which the mass point is shifted.

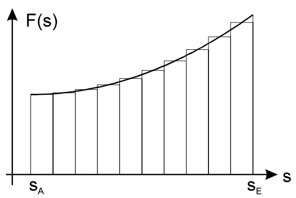


Figure 8.6. Non-constant force

However, if the force is **location dependent** $F = F\,(s)$ (see Fig. 8.6), the path is divided into small intervals $\Delta\,s$. We assume a constant force in each sub-interval. In the $\Delta\,s_i$ interval, the work is then approximated by

$$W_i = F\,(s_i) \cdot \Delta\,s_i.$$

The total work W is the sum of all single contributions

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} F(s_i) \cdot \Delta s_i \,.$$

The exact value of the work done is obtained by going to any fine subdivision ($\Delta s_i \to 0$ or $n \to \infty$):

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} W_i = \lim_{n\to\infty} \sum_{i=1}^{n} F(s_i)\, \Delta s_i = \int_{s_A}^{s_E} F(s)\, ds \,.$$

Work Integral

Arbeitsintegral

Summary: The **work** of a location-dependent force is given by

$$W = \int_{s_A}^{s_E} F(s)\, ds,$$

if $F(s)$ is the force component in path direction. Otherwise $F(s)\,ds$ must be replaced by the scalar product

$$\vec{F}\vec{ds} = \left|\vec{F}\right| \cos\varphi \, ds$$

if φ is the angle between $\vec{F}$ and the direction $\vec{s}$.

Application Example 8.28 (Tension of an Elastic Spring).

If an elastic spring is extended by s from its rest position, a restoring force acts proportionally to the deflection:

$$F(s) = -D \cdot s \qquad \text{(Hook's Law)}.$$

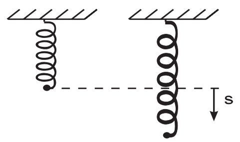


Figure 8.7. Spring-mass system

To deflect a spring from the rest position by the distance s_0, the following work is done:

$$W = \int_0^{s_0} F(s)\, ds = D \int_0^{s_0} s\, ds = \tfrac{1}{2} D\, s_0^2 \,.$$ □

Application Example 8.29 **(Work in the Electrostatic Field).** Two point charges q_1 and q_2 exert a force on each other which is inversely proportional to the square of their distance

$$F = \frac{1}{4\pi\,\varepsilon_0}\,\frac{q_1 \cdot q_2}{r^2}\,.$$

If q_1 is in the source and q_2 is moved from r_1 to r_2, we have to spend the following work:

$$W = \int_{r_1}^{r_2} F\,(r)\;dr = \frac{q_1\,q_2}{4\pi\,\varepsilon_0}\int_{r_1}^{r_2}\frac{1}{r^2}\,dr = \frac{q_1\,q_2}{4\pi\,\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\,. \qquad \square$$

8.6.5 Linear and Quadratic Mean Values

Problem: A two-way rectifier generates $i\,(t) = |i_0\,\sin\,(\omega t)\,|$ from a sinusoidal alternating current with $\omega = \frac{2\pi}{T}$ shown in Fig. 8.8. We're looking for the *linear mean value.*

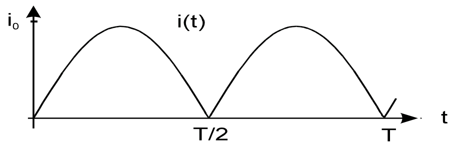


Figure 8.8. Two-way rectifier

Linear Mean Value
Linearer Mittelwert

Definition: *We define the* **Linear Mean Value** *of a function* $y = f\,(x)$ *in interval* $[a,\,b]$ *as the magnitude*

$$\bar{y} = \frac{1}{b-a}\int_a^b f\,(x)\;dx.$$

Remarks:

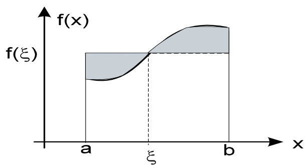


Figure 8.9. Mean value $f\,(\xi)$

According to the mean value theorem of the integral calculus there is always an intermediate value $\xi \in (a,\,b)$, such that $\bar{y} = f\,(\xi)$. Geometrically, the linear mean value means that the area under the curve is replaced by a rectangle of the same area.

Application Example 8.30 **(Linear Mean of a Two-way Rectifier).**

The *linear* mean value of the two-way rectifier (see Fig. 8.8) is calculated using the integral

$$\bar{i} = \frac{1}{T/2} \int_0^{T/2} i_0 \sin(\omega t)\, dt = -\frac{2}{T}\, i_0\, \frac{1}{\omega} \cos(\omega t)\Big|_0^{T/2}$$
$$= -\frac{2}{T}\, i_0\, \frac{T}{2\pi} \left[\cos(\frac{2\pi}{T} \cdot \frac{T}{2}) - 1\right] = \frac{2}{\pi}\, i_0\,.$$

The linear mean value of a sinusoidal alternating current, however, is 0. □

In electrotechnical applications, therefore, the *root mean square value (RMS) is introduced.*

Root Mean Square Value

Quadratischer Mittelwert

Definition: *We define the* **Root Mean Square (RMS) Value** *of a function $y = f(x)$ in interval $[a, b]$ as the magnitude*

$$\bar{y}_q = \left(\frac{1}{b-a} \int_a^b f^2(x)\, dx\right)^{\frac{1}{2}}.$$

Application Example 8.31 **(Effective Value of a AC).**

Effective Value

Effektivwert

The **Effective Value** I_{eff} of an alternating current is the *root mean square value* over a period T:

$$\begin{aligned} I_{eff} &= \left(\frac{1}{T} \int_0^T i^2(t)\, dt\right)^{\frac{1}{2}} = \left(\frac{1}{T} \int_0^T i_0^2 \sin^2(\omega t)\, dt\right)^{\frac{1}{2}} \\ &= \left(\frac{i_0^2}{T} \int_0^T \sin^2(\omega t)\, dt\right)^{\frac{1}{2}} = \left(\frac{i_0^2}{T} \left[\tfrac{1}{2}\, t - \frac{1}{4\omega} \sin(2\,\omega t)\right]_0^T\right)^{\frac{1}{2}} \\ &= \left(\frac{i_0^2}{T} \left[\tfrac{1}{2}\, T - \frac{1}{4\omega} \sin(2\,\omega T)\right]\right)^{\frac{1}{2}} = \frac{i_0}{\sqrt{2}}\,. \end{aligned}$$

Analogously, the effective value of an alternating voltage is $U_{eff} = \frac{U_0}{\sqrt{2}}$. □

8.6.6 Center of Gravity of a Plane Surface

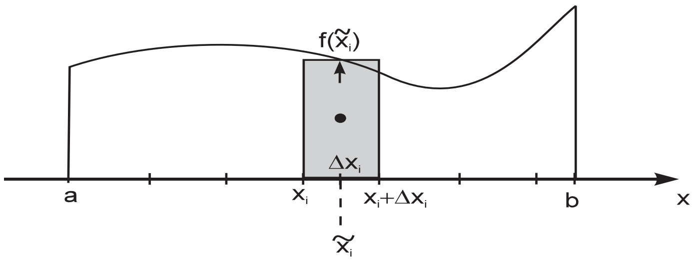


Figure 8.10. Calculation of the center of gravity of a plane surface

If $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ are the coordinates of n mass points with masses $m_1, m_2, \ldots, m_n$, then the coordinates of the *center of gravity* are obtained by

$$x_s = \frac{\sum_{i=1}^{n} m_i\, x_i}{\sum_{i=1}^{n} m_i} \quad \text{and} \quad y_s = \frac{\sum_{i=1}^{n} m_i\, y_i}{\sum_{i=1}^{n} m_i}.$$

To calculate the center of gravity of a plane surface, we assume that the surface is bounded by a curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ (Fig. 8.10). We homogeneously occupy the area with the mass density 1. To calculate the center of gravity we divide the interval $[a, b]$ into n sub-intervals, $a = x_0 < x_1 < \ldots < x_n = b$, select for each subinterval Δx_i an intermediate point $\tilde{x}_i \in [x_i, x_i + \Delta x_i]$ and determine $f(\tilde{x}_i)$. The center of gravity of each rectangle $A_i = \Delta x_i\, f(\tilde{x}_i)$ is

$$x_{s,\,i} = x_i + \frac{1}{2}\Delta x_i \quad , \quad y_{s,\,i} = \frac{1}{2} f(\tilde{x}_i)$$

with the mass $m_i = \frac{A_i}{A} = A_i = \Delta x_i\, f(\tilde{x}_i)$ $(i = 1 \ldots n)$. The coordinates of the center of gravity of the n masses are obtained with coordinates $(x_{s,1}, y_{s,1})$, …, $(x_{s,n}, y_{s,n})$. According to the above formulas we get

$$x_s = \frac{\sum_{i=1}^{n} m_i\, x_{s,\,i}}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \Delta x_i\, f(\tilde{x}_i) \cdot \left(x_i + \frac{1}{2}\Delta x_i\right)}{\sum_{i=1}^{n} \Delta x_i\, f(\tilde{x}_i)}$$

$$y_s = \frac{\sum_{i=1}^{n} m_i\, y_{s,\,i}}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \Delta x_i\, f(\tilde{x}_i) \cdot \frac{1}{2} f(\tilde{x}_i)}{\sum_{i=1}^{n} \Delta x_i\, f(\tilde{x}_i)}$$

For $n \to \infty$, the subtotals merge into the corresponding integrals.

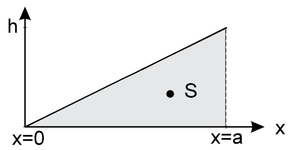
Center of Gravity
Schwerpunkt

Theorem: The coordinates of the **Center of Gravity** $S = (x_s, y_s)$ for the area under a graph $y = f(x)$ between $x = a$ and $x = b$ are

$$x_s = \frac{\int_a^b x\, f(x)\, dx}{\int_a^b f(x)\, dx} \quad \text{and} \quad y_s = \frac{\frac{1}{2}\int_a^b f^2(x)\, dx}{\int_a^b f(x)\, dx}.$$

Examples 8.32:

① The coordinates of the center of gravity for the adjacent triangle below the straight line $y = f(x) = \frac{h}{a}\, x$ are given by



$$x_s = \frac{1}{A}\int_0^a x\, \frac{h}{a}\, x\, dx = \frac{1}{A}\,\frac{h}{a}\left[\frac{x^3}{3}\right]_0^a = \frac{1}{A}\,\frac{h}{3}\, a^2\,,$$

$$y_s = \frac{1}{2\,A}\int_0^a \left(\frac{h}{a}\, x\right)^2 dx = \frac{1}{2\,A}\,\frac{h^2}{a^2}\left[\frac{x^3}{3}\right]_0^a = \frac{1}{2\,A}\,\frac{h^2}{3}\, a\,.$$

With

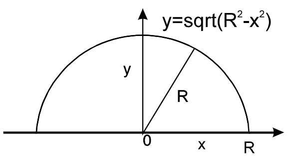
$$A = \int_0^a f(x)\, dx = \int_0^a \frac{h}{a}\, x\, dx = \frac{h}{a}\,\frac{x^2}{2}\bigg|_0^a = \frac{1}{2}\, h\, a$$

results in

$$\boxed{x_s = \tfrac{2}{3}\, a} \quad \text{and} \quad \boxed{y_s = \tfrac{1}{3}\, h.}$$

② We calculate the center of gravity coordinates of the adjacent semicircle with radius R: Due to symmetry reasons the center of gravity is on the y-axis, such that $x_s = 0$. For the y-coordinate of the center of gravity it yields

$$\begin{aligned} y_s &= \tfrac{1}{2\,A}\int_{-R}^{R} y^2\, dx \\ &= \tfrac{1}{2\,A}\int_{-R}^{R} \left(R^2 - x^2\right)\, dx \\ &= \tfrac{1}{2\,A}\left[R^2\, x - \tfrac{1}{3}\, x^3\right]_{-R}^{R} = \tfrac{1}{2\,A}\,\tfrac{4}{3}\, R^3\,. \end{aligned}$$

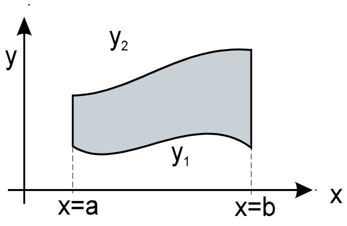


With $A = \frac{\pi}{2}\, R^2$ follows altogether $\quad y_s = \dfrac{4}{3\pi}\, R.$

③ The coordinates of the center of gravity of the area A, which is limited by two functions $y_2 = f(x)$ and $y_1 = g(x)$ with $f \geq g$ as well as the straight lines $x = a$ and $x = b$, are given by the difference of the single centers of gravity:

$$x_s = \frac{1}{A} \int_a^b x \, [f(x) - g(x)] \, dx$$

$$y_s = \frac{1}{2A} \int_a^b \left[f^2(x) - g^2(x)\right] \, dx$$



and

$$A = \int_a^b [f(x) - g(x)] \, dx. \qquad \square$$

8.7

8.7 More Applications of Integral Calculus

8.7.1 Averaging property

An important property of the integral is its smoothing behavior, because

$$\frac{1}{(b-a)} \int_a^b f(x) \, dx$$

is the mean value a function in the interval $[a, b]$. This property is used when interpreting measurement results, as these are usually distorted by noise.

Application Example 8.33 . We consider the function

$$f := x^2 \left(1 + \frac{1}{20} \sin(200\,x)\right) + \frac{1}{20} \cos(50\,x)$$

which corresponds on average to a x^2-function, but is overlaid with high-frequency noise.

Linear averaging with a suitable interval length $h = 0.1$ results in a smooth curve:
If h is selected too small, oscillations are still observed (e.g. for $h = 0.05$); if h is too large, the resulting function becomes angular (e.g. for $h = 0.5$). The appropriate h is oriented to the occurring interference frequencies. Here,

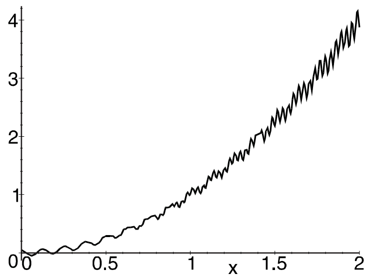


Figure 8.11. Non smooth function

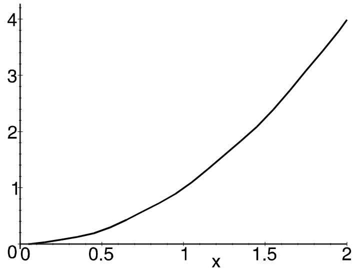


Figure 8.12. Smooth function

the smallest frequency is $\omega = 50 = 2\pi/T$ corresponding to a period of $T = 2\pi/50 = 0.125$. The suitable h is therefore about 0.1. However, in practice the interference frequencies are not known. To reconstruct them from the signal, methods of a Fourier analysis must be used. □

8.7.2 Arc length

The *arc length* of a curve $\overset{\frown}{AB}$ is calculated by first dividing the interval $[a, b]$ into n subintervals

$$a = x_0 < x_1 < x_2 < \ldots < x_n = b.$$

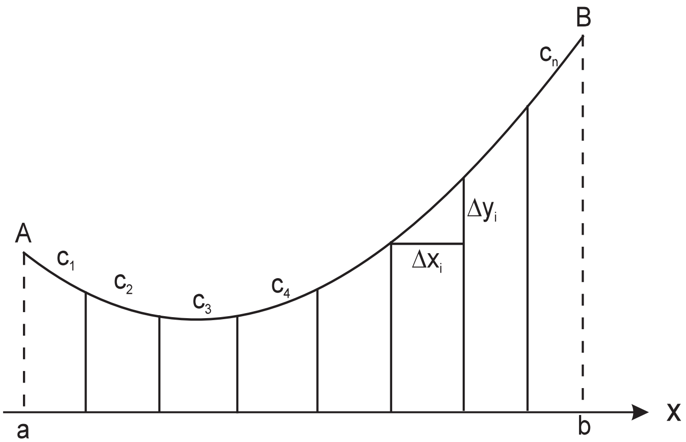


Figure 8.13. Arc length of a curve

For each subinterval the curve is replaced by a straight line $c_1, c_2, \ldots, c_n$ with the individual lengths

$$|c_i| = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i.$$

The total length of all pieces is

$$S_n = \sum_{i=1}^{n} |c_i| = \sum_{i=1}^{n} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \cdot \Delta x_i\,.$$

By refining the decomposition of the interval $[a,\, b]$ with $n \to \infty$, the graph of f is approximated precisely by the stretch. If the limit value $\lim_{n\to\infty} S_n$ exists, f is called *rectifiable* and the limit value is the *arc length of the graph*.

To calculate the limit value, the mean value theorem of the differential calculation is used. According to this theorem, there is for each interval $\tilde{x}_i \in [x_i,\, x_i + \Delta x_i]$, so that $\frac{\Delta y_i}{\Delta x_i} = f'(\tilde{x}_i)$. For $n \to \infty$ then $\Delta x_i \to 0$ and $\frac{\Delta y_i}{\Delta x_i} \to f'(x_i)$. Thus, the arc length results in

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + (f'(\tilde{x}_i))^2} \cdot \Delta x_i = \int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$

Arc Length
Bogenlänge

Theorem: Let f be a function that is continuously differentiable on the interval $[a,\, b]$. Then the **arc length** S of the function graph of $y = f(x)$ is valid between $x = a$ and $x = b$

$$S = \int_a^b \sqrt{1 + (f'(x))^2}\, dx = \int_a^b \sqrt{1 + (y')^2}\, dx.$$

Example 8.34. Find the arc length of the function $y = \cosh(x)$ in the range from $x = 0$ to $x = 1$:

The derivative of $y = \cosh(x)$ is $y' = \sinh(x)$. This gives the integrand to

$$\hookrightarrow \sqrt{1 + (y')^2} = \sqrt{1 + \sinh^2(x)} = \cosh(x)\,,$$

$\cosh^2(x) - \sinh^2(x) = 1$. So the arc length is given by

$$S = \int_0^1 \cosh(x)\; dx = \sinh(x)|_0^1 = \sinh(1) = 1.175\,.$$

□

Example 8.35. Calculation of the arc length of a quarter circle (see Figure 8.14).

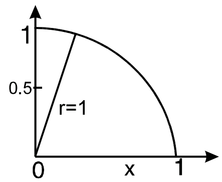


Figure 8.14. Circle

The circle equation is $x^2 + y^2 = 1$. We solve this equation for

$$y = f(x) = \sqrt{1 - x^2}$$

and obtain the derivative

$$y' = \frac{1}{2}\frac{1}{\sqrt{1-x^2}} \cdot (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

The integrand for calculating the arc length thus results in

$$\sqrt{1+y'^2} = \sqrt{1 + \frac{x^2}{1-x^2}} = \sqrt{\frac{1-x^2+x^2}{1-x^2}} = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}}.$$

To calculate the integral

$$S = \int_0^1 \sqrt{1+y'^2}dx = \int_0^1 (1-x^2)^{-\frac{1}{2}}dx$$

a substitution is chosen

$$x = \sin(t).$$

Because then

$$(1-x^2)^{-\frac{1}{2}} = (1-\sin^2(t))^{-\frac{1}{2}} = (\cos^2(t))^{-\frac{1}{2}} = (\cos(t))^{-1}$$

and

$$\frac{dx}{dt} = \cos(t) \Rightarrow dx = \cos(t)dt.$$

Inserted into the integral, we get the results taking the limits into account $t_u = \arcsin(x_u) = \arcsin(0) = 0$ and $t_o = \arcsin(x_o) = \arcsin(1) = \pi/2$:

$$S = \int_0^1 (1-x^2)^{-1/2}dx = \int_0^{\pi/2} \frac{1}{\cos(t)}\cos(t)dt = [t]_0^{\pi/2} = \frac{\pi}{2}. \qquad \square$$

Example 8.36. The arc length of $y = x^2$ in the range from $x = 0$ to $x = 2$ is

$$\int_0^2 \sqrt{1+4\,x^2}\,dx = \sqrt{17} + \frac{1}{4}\ln(\,4+\sqrt{17}\,) = 4.646783762 \qquad \square$$

8.8

8.8 Problems on Integral Calculus

8.1 Take the function $f(x) = \sqrt{x}$ in the $x \in [0,\, 2]$ range and calculate for $n = 10$ the right sum.

8.2 The following integrals are to be found:

a) $\int x^5\, dx$ b) $\int \frac{dx}{x^2}$ c) $\int \sqrt[3]{z}\, dz$

d) $\int \frac{dx}{\sqrt[3]{x^2}}$ e) $\int \left(2\, x^2 - 5x + 3\right)\, dx$ f) $\int (1-x)\,\sqrt{x}\, dx$

8.3 Calculate the following definite integrals:

a) $\int_0^{\frac{\pi}{2}} \sin x\, dx$ b) $\int_{\frac{1}{2}\pi}^{\frac{3}{2}\pi} (2\, x + \sin x - \cos x)\, dx$ c) $\int_1^a \frac{1}{x}\, dx$

8.4 Use integration by parts to determine the following integrals:

a) $\int x\, \cos x\, dx$ b) $\int \sin x\, \cos x\, dx$ c) $\int x^2\, \sin x\, dx$

d) $\int x^2\, \ln x\, dx$ e) $\int x\, e^x\, dx$ f) $\int x^2\, e^x\, dx$

8.5 Determine the following integrals by substitution:

a) $\int \frac{1}{x+2}\, dx$ b) $\int \frac{x}{x^2-1}\, dx$ c) $\int \frac{x^2}{1-2\,x^3}\, dx$

d) $\int (3\, s + 4)^8\, ds$ e) $\int \sin(\omega t + \varphi)\, dt$ f) $\int \cos(3\, t)\, dt$

g) $\int e^{-x}\, dx$ h) $\int \frac{\sin t}{\cos t}\, dt$ i) $\int \frac{e^x + x\, e^x}{x\, e^x}\, dx$

j) $\int \sin x\, \cos x\, dx$ k) $\int \sqrt{4 + 3\, x}\, dx$

8.6 Show the validity of the following expressions

a) $\int e^{-x}\, (1-x)\, dx = x\, e^{-x} + C$

b) $\int \frac{\sqrt{x^2-4}}{x}\, dx = \sqrt{x^2-4} - 2\, \arccos\left(\frac{2}{x}\right) + C$

c) $\int \cos(x)\, e^{\sin(x)}\, dx = e^{\sin(x)} + C$

d) $\int \cos(3x) \cdot \sin(3x)\, dx = \frac{1}{6}\, \sin^2(3\, x) + C$

8.7 a) Solve the integral $\int \frac{2-x}{1+\sqrt{x}}\, dx$ with the substitution $u = 1 + \sqrt{x}$.

b) Solve the integral $\int x\, \sqrt{1-x^2}\, dx$ with the substitution $x = \sin u$.

8.8 Calculate the following integrals

a) $\int \frac{x^2}{\sqrt{1+x^3}}\, dx$ b) $\int (5\, x + 12)^{\frac{1}{2}}\, dx$ c) $\int \sqrt[3]{1-t}\, dt$

d) $\int_0^{\pi} \cos^3 x \cdot \sin x\, dx$ e) $\int \frac{\arctan z}{1+z^2}\, dz$ f) $\int \frac{2\,x+6}{x^2+6\,x-12}\, dx$

g) $\int \frac{dx}{x\, \ln x}$ h) $\int x \cdot \sin\left(x^2\right)\, dx$ i) $\int \frac{3\,x^2-2}{2\,x^3-4\,x+2} dx$

j) $\int_{-1}^{1} \frac{t}{\sqrt{1+t^2}}\, dt$ k) $\int_0^{\frac{\pi}{2}} \sin\left(3\, t - \frac{\pi}{4}\right)\, dt$ l) $\int_{-1}^{1} \frac{5+x}{5-x}\, dx$

m) $\int x^2\, e^{x^3-2}\, dx$ n) $\int \frac{\tan(z+5)}{\cos^2(z+5)}\, dz$ o) $\int \frac{\sqrt{4-x^2}}{x^2}\, dx$

8.9 Solve the following integrals by integration by parts

a) $\int x \ln x \, dx$ b) $\int x \cos x \, dx$ c) $\int \ln t \, dt$

d) $\int x \sin(3x) \, dx$ e) $\int \arctan x \, dx$ f) $\int \sin^2(\omega t) \, dt$

g) $\int e^x \cos x \, dx$ h) $\int x^2 e^{-x} \, dx$

8.10 Solve the following integrals by partial fractional decomposition:

a) $\int \frac{1}{x^2-a^2} \, dx$ b) $\int \frac{4x^3}{x^3+2x^2-x-2} \, dx$ c) $\int \frac{3z}{z^3+3z^2-4} \, dz$

d) $\int \frac{4x-2}{x^2-2x-63} \, dx$ e) $\int \frac{2x+1}{x^3-6x^2+9x} \, dx$

8.11 Calculate the following integrals

a) $\int \frac{\sqrt{\ln x}}{x} \, dx$ b) $\int \cot x \, dx$ c) $\int x \cosh x \, dx$

d) $\int \sin x \, e^{\cos x} \, dx$ e) $\int \frac{x^3}{(x^2-1)(x+1)} \, dx$ f) $\int \frac{x-4}{x+1} \, dx$

g) $\int \frac{(\ln x)^3}{x} \, dx$ h) $\int \frac{12x^2}{2x^3-1} \, dx$ i) $\int x \cdot \arctan x \, dx$

8.12 Determine $\int \ln(x + \sqrt{1+x^2}) \, dx$ by first performing the substitution $u^2 = 1 + x^2$ and then integrating it by parts.

8.13 Calculate the indefinite integral $\int \frac{1}{\sqrt{1+\sqrt{x}}} dx$ by the substitution $u^2 = 1+\sqrt{x}$.

8.14 Perform a partial fractional decomposition and then integrate

a) $\int \frac{x^4-x^3-x-1}{x^3-x^2} \, dx$ b) $\int \frac{x^4-x^3+3x}{x^2(x+2)(x-3)} \, dx$ c) $\int \frac{x^4-x^3+3x^2-2x+1}{x^3-x^2-x+1}$

8.15 Calculate the following definite integrals ($n, m \in \mathbb{N}$) by using the addition theorems for sine and cosine.

a) $\int_0^{2\pi} \sin(nx) \, dx$ b) $\int_0^{2\pi} \cos(nx) \, dx$

c) $\int_0^{2\pi} \cos(nx) \cos(mx) \, dx$ for $m = n$ and for $m \neq n$

d) $\int_0^{2\pi} \sin(nx) \sin(mx) \, dx$ for $m = n$ and for $m \neq n$

e) $\int_0^{2\pi} \sin(nx) \cos(mx) \, dx$

8.16 Determine the value of the improper integrals

a) $\int_2^{\infty} x^{-2} \, dx$ b) $\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx$ c) $\int_0^{\infty} \left(3e^{-2x} - e^{-x}\right) dx$

d) $\int_0^{\infty} e^{at} e^{-st} dt$ e) $\int_0^{\infty} \cos(at) \, e^{-st} dt$ e) $\int_0^{\infty} t^n \, e^{-st} dt$

8.17 Calculate the area between the parabola
$f(x) = x^2 - 2x - 1$ and the straight line $g(x) = 3x - 1$.

Additional Problems on Integral Calculus

8.a1 Calculate the average power of a sinusoidal alternating current

$$\bar{P} = \frac{1}{T} \int_0^T P(t)\, dt,$$

if $P(t) = U(t) \cdot I(t) = u_0 \sin(\omega t) \cdot i_0 \sin(\omega t + \varphi)$.

8.a2 Three mean values are defined for an alternating current $I(t)$ with period T:

$$I_{eff} := \sqrt{\frac{1}{T} \int_0^T I^2(t)\, dt} \qquad \text{(Effective value)}$$

$$\hat{I} := \frac{1}{T} \int_0^T I(t)\, dt \qquad \text{(Linear average value)}$$

$$|\hat{I}| := \frac{1}{T} \int_0^T |I(t)|\, dt \qquad \text{(Rectified value)}$$

Calculate these quantities for
a) for $I(t) = I_0 \sin(\frac{2\pi}{T} t)$
b) for a sawtooth current these three average values.

8.a3 Determine the arc length and the curvature of the curve $y = x^3$ between $x = 0$ and $x = 5$.

8.a4 Calculate the arc length of the chain line $y = a \cosh\left(\frac{x}{a}\right)$ between $x = 0$ and $x = b$.

8.a5 Calculate the lengths and curvatures of the curves
a) $y = \frac{1}{2} x^2$, $-1 \leq x \leq 1$
b) $y = \cosh(x)$, $-a \leq x \leq a$.

8.a6 Show that the volume of a truncated cone is

$$V = \frac{1}{3} \pi h \left(R^2 + R r + r^2\right)$$

by dividing the linear equation by the points $(0, r)$ and (h, R) and rotating the graph around the x axis.

8.a7 Show that the arc length of a semicircle with radius $r = 1$ π.

8.a8 Determine the arc length of the graph of the function $f(x) = \frac{2}{3} x^{\frac{3}{2}}$ between the points $(0, 0)$ and $(3, f(3))$. What is the curvature of the curve? What is its volume of rotation?

8.a9 Determine the volume of the body, which is generated by rotating the curve $y = a x^2$ $(x \in [0, r])$ around the y axis.

8.a10 Create a procedure to calculate the center of gravity coordinates of plane surfaces represented by a graph $y = f(x)$ of the x axis and $x = a$, $x = b$ are limited.

8.a11 Calculate the center of gravity coordinates of the area between the graph $f(x) = h$ and $g(x) = \frac{h}{a^2} x^2$ for $x \in [0, a]$.