

Contents

1	Some elements of stochastic analysis	1
1.1	Stochastic processes	1
1.1.1	Filtration and processes	1
1.1.2	Stopping times	3
1.1.3	Brownian motion	5
1.1.4	Martingales, semimartingales	6
1.2	Stochastic integral and applications	12
1.2.1	Stochastic integral with respect to a continuous semimartingale	12
1.2.2	Itô process	16
1.2.3	Itô's formula	17
1.2.4	Martingale representation theorem	18
1.2.5	Girsanov's theorem	18
1.3	Stochastic differential equations	22
1.3.1	Strong solutions of SDE	22
1.3.2	Estimates on the moments of solutions to SDE	24
1.3.3	Feynman-Kac formula	25
2	Stochastic optimization problems. Examples in finance	27
2.1	Introduction	27
2.2	Examples	28
2.2.1	Portfolio allocation	28
2.2.2	Production-consumption model	29
2.2.3	Irreversible investment model	30
2.2.4	Quadratic hedging of options	31
2.2.5	Superreplication cost in uncertain volatility	31
2.2.6	Optimal selling of an asset	32
2.2.7	Valuation of natural resources	32
2.3	Other optimization problems in finance	32
2.3.1	Ergodic and risk-sensitive control problems	32
2.3.2	Superreplication under gamma constraints	33

2.3.3	Robust utility maximization problem and risk measures	33
2.3.4	Forward performance criterion	34
2.4	Bibliographical remarks	34
3	The classical PDE approach to dynamic programming	37
3.1	Introduction	37
3.2	Controlled diffusion processes	37
3.3	Dynamic programming principle	40
3.4	Hamilton-Jacobi-Bellman equation	42
3.4.1	Formal derivation of HJB	42
3.4.2	Remarks and extensions	45
3.5	Verification theorem	47
3.6	Applications	51
3.6.1	Merton portfolio allocation problem in finite horizon	51
3.6.2	Investment-consumption problem with random time horizon	53
3.6.3	A model of production-consumption on infinite horizon	55
3.7	Example of singular stochastic control problem	58
3.8	Bibliographical remarks	59
4	The viscosity solutions approach to stochastic control problems	61
4.1	Introduction	61
4.2	Definition of viscosity solutions	61
4.3	From dynamic programming to viscosity solutions of HJB equations	64
4.3.1	Viscosity properties inside the domain	64
4.3.2	Terminal condition	69
4.4	Comparison principles and uniqueness results	75
4.4.1	Classical comparison principle	76
4.4.2	Strong comparison principle	77
4.5	An irreversible investment model	82
4.5.1	Problem	82
4.5.2	Regularity and construction of the value function	83
4.5.3	Optimal strategy	88
4.6	Superreplication cost in uncertain volatility model	89
4.6.1	Bounded volatility	90
4.6.2	Unbounded volatility	91
4.7	Bibliographical remarks	94
5	Optimal switching and free boundary problems	95
5.1	Introduction	95
5.2	Optimal stopping	95
5.2.1	Dynamic programming and viscosity property	96
5.2.2	Smooth-fit principle	99
5.2.3	Optimal strategy	101
5.2.4	Methods of solution in the one-dimensional case	103
5.2.5	Examples of applications	104

5.3	Optimal switching	107
5.3.1	Problem formulation	108
5.3.2	Dynamic programming and system of variational inequalities	109
5.3.3	Switching regions	114
5.3.4	The one-dimensional case	116
5.3.5	Explicit solution in the two-regime case	119
5.4	Bibliographical remarks	137
6	Backward stochastic differential equations and optimal control	139
6.1	Introduction	139
6.2	General properties	139
6.2.1	Existence and uniqueness results	139
6.2.2	Linear BSDE	141
6.2.3	Comparison principles	142
6.3	BSDE, PDE and nonlinear Feynman-Kac formulae	143
6.4	Control and BSDE	147
6.4.1	Optimization of a family of BSDEs	147
6.4.2	Stochastic maximum principle	149
6.5	Reflected BSDEs and optimal stopping problems	152
6.5.1	Existence and approximation via penalization	154
6.5.2	Connection with variational inequalities	159
6.6	Applications	162
6.6.1	Exponential utility maximization with option payoff	162
6.6.2	Mean-variance criterion for portfolio selection	165
6.7	Bibliographical remarks	169
7	Martingale and convex duality methods	171
7.1	Introduction	171
7.2	Dual representation for the superreplication cost	172
7.2.1	Formulation of the superreplication problem	172
7.2.2	Martingale probability measures and no arbitrage	173
7.2.3	Optional decomposition theorem and dual representation for the superreplication cost	174
7.2.4	Itô processes and Brownian filtration framework	177
7.3	Duality for the utility maximization problem	181
7.3.1	Formulation of the portfolio optimization problem	181
7.3.2	General existence result	181
7.3.3	Resolution via the dual formulation	183
7.3.4	The case of complete markets	195
7.3.5	Examples in incomplete markets	197
7.4	Quadratic hedging problem	199
7.4.1	Problem formulation	199
7.4.2	The martingale case	200
7.4.3	Variance optimal martingale measure and quadratic hedging numéraire	201

7.4.4	Problem resolution by change of numéraire	206
7.4.5	Example	210
7.5	Bibliographical remarks	212
A	Complements of integration	213
A.1	Uniform integrability	213
A.2	Essential supremum of a family of random variables	215
A.3	Some compactness theorems in probability	215
B	Convex analysis considerations	217
B.1	Semicontinuous, convex functions	217
B.2	Fenchel-Legendre transform	218
B.3	Example in $\mathbb{R}$	219
	References	223
	Index	231